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New types of domination to characterize the preservation of T-subgroups under aggregation

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ABSTRACT

This study builds upon results concerning the preservation of fuzzy structures by aggregation operators. We analyze when an aggregation operator preserves the structure of T-subgroup in the last cases remaining unresolved in the characterization of such operators. In order to characterize these operators for different groups, relaxed forms of domination are introduced and their properties and interconnections are studied. We also include a thorough review of the features that an aggregation operator must fulfill to preserve T-subgroups of an arbitrary group.

1. Introduction

In fuzzy set theory, aggregation operators play a fundamental role in combining fuzzy membership values into a single representative output. The selection of the aggregation function depends on the specific application. Some examples of these applications include decision making, pattern recognition, classification and fuzzy control.

In this context, the preservation of various fuzzy structures has been a significant area of study in recent years, as evidenced by the large number of works published on the subject (see for instance [6–8,16,19,23–25,27,31]). The primary objective of these papers is to identify conditions that ensure the preservation of certain properties of particular types of fuzzy sets during aggregation. There are two distinct complementary methods of defining this aggregation, each with its own interpretation and purpose (refer to [14,26,33]). They are respectively called aggregation *on sets* and aggregation *on products*. The introductory section of [3] provides a comprehensive discussion of the use of these definitions in the literature. Since both types of aggregation are widely used and accepted, it is interesting to analyze the necessary and sufficient conditions for an operator to preserve a property under both formalisms. In some cases, one fuzzy property is preserved for both definitions, such as with fuzzy transitivity (see [14,33]). However, operators that preserve fuzzy quasi-metrics differ depending on whether the aggregation is done *on products* or *on sets* (see [26] for more details). Consequently, when studying the preservation of structures in fuzzy set theory, it is convenient to compare both definitions.

T-subgroups are closely related to invariant T-indistinguishability operators ([11,13,17]), which are relevant in a wide range of applications, as it is dealt in references [9,10,30,38]. This paper concludes the study of the aggregation of fuzzy T-subgroups, which has been an intense research area in recent years. In [4], the authors gave the conditions for a set of T_M-subgroups to become again

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a T_M-subgroup after being aggregated on sets. For the case of an arbitrary t-norm, it was necessary to distinguish cases based on the type of group G over which the T-subgroups are defined. The lattice of subgroups of G, denoted by $Lat(G)$, is the key factor. In [3], we analyzed the relationships between the two definitions of aggregation of T-subgroups when $Lat(G)$ is not a chain. This same problem when G belongs to the set $C = \{G \mid G \text{ is a group and } Lat(G) \text{ is a chain}\}$ was considered only for T-subgroups defined with a specific t-norm, the minimum t-norm. The paper [39] presents necessary and sufficient conditions for an aggregation operator to preserve T-subgroups *on products* with an arbitrary t-norm when we have an ambient group G belonging to the class C. Nevertheless, the preservation of T-subgroups *on sets* when G is in C has not yet been considered for an arbitrary t-norm and will be addressed in this manuscript. Therefore, we can establish the next list of problems in which the topic of the preservation of T-subgroups is decomposed. The works where they are studied are also provided:

- Preservation on products when the group is not in C. It is studied in [3].
- Preservation on sets when the group is not in C. It is studied in [3].
- Preservation on products when the group is in C. It is studied in [39].
- Preservation on sets when the group is in C. It will be studied in this article.

Thus, the present work addresses this last important problem and fills this gap. This completes the study of the characterization of aggregation operators that preserve T-subgroups for both definitions. It is shown that while the aggregation operators preserving T-subgroups *on products* also preserve T-subgroups *on sets*, not all aggregations preserving T-subgroups *on sets* do so *on products*. Therefore, although it is easy to obtain sufficient conditions for the preservation of T-subgroups under aggregation on sets using previous results on products, the conditions thus obtained are not necessary. The characterizations given in this work provide the corresponding necessary and sufficient conditions and consequently a method to classify the cases when an aggregation preserves T-subgroups *on sets* but not *on products*. To this end, new restricted forms of domination will be introduced and studied, since the usual concept of domination is, in general, too strong a requirement. This manuscript addresses the task of identifying how this new types of domination characterize the preservation of T-subgroups *on sets*. Furthermore, the findings presented here offer a comprehensive and conclusive answer to the issue of characterizing the preservation of T-subgroups.

This paper has the following structure: Section 2 sets the ground for the subsequent discussion by introducing the necessary definitions. Some concepts that play a key role in the preservation of T-subgroups are introduced and studied in Section 3 as well as their interconnections. Section 4 provides some background and discusses the current state of the art regarding the aggregation of T-subgroups. In Section 5, we study the conditions that are both necessary and sufficient for the preservation of T-subgroups *on sets* when the ambient group G belongs to C. In Section 6 we compare the results obtained in this study with those obtained in [39] regarding the preservation of T-subgroups *on products*. Finally, Section 7 provides a general overview and introduces potential directions for future studies.

2. Preliminaries

In this section we present some preliminary concepts that will be used in the paper. In the first place, we need to define the operators that aggregate information.

Definition 2.1 *([12])*. Let $\boldsymbol{A} : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ be a function. $\boldsymbol{A}$ is called aggregation operator or aggregation function if:

A1. For all $\boldsymbol{x} = (x_1, \dots, x_n)$, $\boldsymbol{y} = (y_1, \dots, y_n) \in [0,1]^n$ such that $x_i \le y_i$ $\forall i \in \{1, \dots, n\}$, we have $\boldsymbol{A}(x_1, \dots, x_n) \le \boldsymbol{A}(y_1, \dots, y_n)$.
A2. $\boldsymbol{A}(\boldsymbol{0}) = \boldsymbol{A}(0, \dots, 0) = 0$ and $\boldsymbol{A}(\boldsymbol{1}) = A(1, \dots, 1) = 1$.
A3. $\boldsymbol{A}(x) = x$ for all $x \in [0,1]$.

Each aggregation operator $\boldsymbol{A}$ can be represented as a collection of n-ary aggregation operators $A_{(n)} : [0,1]^n \to [0,1]$ satisfying A1 and A2 if $n \ge 2$ and additionally A3 if $n = 1$. In order to shorten some of the proofs, we will use the notation A instead of $A_{(n)}$ if the context is clear enough.

Definition 2.2 *([33])*. An aggregation operator $\mathbf{A}$ is said to be associative if for all $n, m \in \mathbb{N}$ and for all $x_1, \dots, x_n, y_1, \dots, y_m \in [0,1]$:

$$\mathbf{A}(x_1, \dots, x_n, y_1, \dots, y_m) = \mathbf{A}(\mathbf{A}(x_1, \dots, x_n), \mathbf{A}(y_1, \dots, y_m)).$$

A triangular norm is a special type of aggregation operator that is a key concept to define a wide variety of fuzzy structures.

Definition 2.3 *([21])*. A triangular norm, t-norm for short, is a binary operation $T : [0,1]^2 \to [0,1]$ which is commutative, associative, monotonically increasing and such that $T(x,1) = x$ for all $x \in [0,1]$.

Example 2.4. Some of the most important and used t-norms are:

1. $T_M(x,y) = \min\{x,y\}$. (Minimum t-norm)

2. $T_D(x,y) = \begin{cases} 0 & \text{if } \max\{x,y\} < 1, \\ \min\{x,y\} & \text{if } \max\{x,y\} = 1. \end{cases}$ (Drastic t-norm)
3. $T_P(x,y) = xy$. (Product t-norm)
4. $T_L(x,y) = \max\{x+y-1, 0\}$. (Łukasiewicz t-norm)

We recall now some notation and properties regarding t-norms that will have an impact later on.

Definition 2.5 *([20]).* Let T be a t-norm, $x \in [0,1]$ and $k \in \mathbb{N}$. We define $x_T^{(k)}$ as:

$$x_T^{(k)} = \begin{cases} x & \text{if } k = 1, \\ T(x_T^{(k-1)}, x) & \text{if } k \neq 1. \end{cases}$$

Remark 2.6. Note that the sequence $\{x_T^{(k)}\}_{k\in\mathbb{N}}$ is non-increasing as a consequence of the monotonicity of T.

Lemma 2.7 *([5]). Given a t-norm T and a number $k \in \mathbb{N}$ greater than or equal to 2, we have that $x_T^{(k)} = T(x_T^{(k_1)}, x_T^{(k_2)})$ for all $x \in [0,1]$ and $k_1, k_2 \in \mathbb{N}$ such that $k_1 + k_2 = k$.*

We are now ready to define T-subgroups according to Anthony and Sherwood in [1,2]. From now on, X will be a non-empty set and $\mu : X \to [0,1]$ a fuzzy set. Moreover, $[0,1]^X$ will denote the set of all fuzzy sets in X.

Definition 2.8. Let G be a group, $\mu \in [0,1]^G$ a fuzzy subset of G and T a t-norm. μ is called fuzzy T-subgroup of G if:

G1. $\mu(e) = 1$ where $e \in G$ denotes the neutral element.
G2. $\mu(x) = \mu(x^{-1})\ \forall x \in G$.
G3. $\mu(xy) \geq T(\mu(x), \mu(y))\ \forall x, y \in G$.

The following results provide valuable information about different types of T-subgroups.

Lemma 2.9 *([5]). Let μ be a T-subgroup of a group G and $k \in \mathbb{N}$. Then $\mu(a^k) \geq \mu(a)_T^{(k)}$ for all $a \in G$.*

The following theorem provides a useful alternative description of the T-subgroups of a cyclic group. Note that $[a]$ denotes the integer part of a real number a.

Theorem 2.10 *([39]). Let $\mu \in [0,1]^G$ with $G = \langle g \rangle$ a cyclic group with order $r \in \mathbb{N}$. Then, μ is a T-subgroup of G if and only if it can be expressed as:*

$$\mu(z) = \begin{cases} 1 & \text{if } z = e, \\ \alpha_1 & \text{if } z \in \{g, g^{-1}\}, \\ \vdots & \qquad \vdots \\ \alpha_{\left[\frac{r}{2}\right]} & \text{if } z \in \{g^{\left[\frac{r}{2}\right]}, g^{-\left[\frac{r}{2}\right]}\}, \end{cases} \tag{1}$$

with $\alpha_1, \dots, \alpha_{\left[\frac{r}{2}\right]} \in [0,1]$ and $\alpha_u \geq T(\alpha_j, \alpha_k)$ for all:

$$(u,j,k) \in D_r = \{(x,y,z) \in \left\{1, \dots, \left[\tfrac{r}{2}\right]\right\}^3 \mid x \in \{y+z, z-y, r-y-z\}\}.$$

Remark 2.11. Some clarification is needed for the previous theorem. A cyclic group can be described in terms of a generator g as the set of all its powers $\{g, g^2, \dots, g^{r-1}, g^r = e\}$. To ensure the fulfillment of the axiom G2, μ takes the same value for g^s and its inverse $g^{-s} = g^{r-s}$ for every $1 \leq s \leq \frac{r}{2} = \left[\frac{r}{2}\right]$ if r is even and for every $1 \leq s \leq \frac{r-1}{2} = \left[\frac{r}{2}\right]$ if r is odd. Thus, all powers of g are included in the expression (1). Additionally, the construction of the set D_r is a consequence of the axiom G3.

As a consequence of Theorem 2.10, we can state an auxiliary lemma that will be used in different proofs in the manuscript.

Lemma 2.12 *([39]). Let T be a t-norm and $G = \langle g \rangle$ a group such that $|G| = r > 5$. Given $x, y \in [0,1]$ and $v \in \left\{2, \dots, \left[\frac{r}{2}\right] - 1\right\}$ with $x \geq x_T^{(v-1)} \geq y \geq x_T^{(v)} \geq x_T^{\left(\left[\frac{r}{2}\right]-1\right)}$. Then, the fuzzy set:*

$$\mu(z)=\begin{cases} 1 & \text{if } z=e,\\ x_T^{(u)} & \text{if } z\in\{g^u,g^{r-u}\} \text{ with } 1\le u<v,\\ y & \text{if } z\in\{g^u,g^{r-u}\} \text{ with } u=v,\\ \alpha_u & \text{if } z\in\{g^u,g^{r-u}\} \text{ with } v+1\le u\le\left[\frac{r}{2}\right]\end{cases} \tag{2}$$

where in the last case α_u can be any number in the interval $[T(x,y),y]$, is a T-subgroup.

Finally, we define the aggregation of fuzzy T-subgroups, both on products and on sets as in [3,39].

Definition 2.13. Let $\boldsymbol{A}:\bigcup_{n\in\mathbb{N}}[0,1]^n\to[0,1]$, be an aggregation operator, T a t-norm and $n\in\mathbb{N}$. Given n fuzzy T-subgroups $\mu_1,\dots,\mu_n$ of a group G, we denote by $\boldsymbol{\mu}$ the map $\boldsymbol{\mu}:G\to[0,1]^n$, with $\boldsymbol{\mu}(x)=(\mu_1(x),\dots,\mu_n(x))$ for x in G and we define the **aggregation of fuzzy T-subgroups on sets** as $A\circ\boldsymbol{\mu}$, where:

$$A\circ\boldsymbol{\mu}(x)=A(\mu_1(x),\dots,\mu_n(x)).$$

We will say that **A preserves the structure of T-subgroup on sets** if and only if $A\circ\boldsymbol{\mu}$ is a T-subgroup for any $\boldsymbol{\mu}$ as above.

In the same way $\tilde{\boldsymbol{\mu}}$ denotes $\tilde{\boldsymbol{\mu}}:\prod_{i=1}^n G\to[0,1]^n$ with $\tilde{\boldsymbol{\mu}}(\boldsymbol{x})=(\mu_1(x_1),\dots,\mu_n(x_n))$ for $\boldsymbol{x}=(x_1,\dots,x_n)$ in $\prod_{i=1}^n G$. We define the **aggregation of fuzzy T-subgroups on products** as $A\circ\tilde{\boldsymbol{\mu}}$, where:

$$A\circ\tilde{\boldsymbol{\mu}}(\boldsymbol{x})=A(\mu_1(x_1),\dots,\mu_n(x_n)).$$

As before, we state that **A preserves the structure of T-subgroup on products** if and only if $A\circ\tilde{\boldsymbol{\mu}}$ is a T-subgroup for any $\tilde{\boldsymbol{\mu}}$ as above.

The next section will provide different properties that are key in the study of the preservation of T-subgroups with both definitions.

3. Introducing new types of domination

The dominance relation is a fundamental property that frequently arises in certain fields of fuzzy set theory as the preservation of fuzzy structures by aggregation.

Definition 3.1 *([33]).* Consider an n-ary aggregation operator $A_{(n)}$ and an m-ary aggregation operator $B_{(m)}$. We say that $A_{(n)}$ dominates $B_{(m)}$ if for all $x_{i,j}\in[0,1]$ with $i\in\{1,\dots,m\}$ and $j\in\{1,\dots,n\}$, the following property holds:

$$B_{(m)}(A_{(n)}(x_{1,1},\dots,x_{1,n}),\dots,A_{(n)}(x_{m,1},\dots,x_{m,n}))$$
$$\le A_{(n)}(B_{(m)}(x_{1,1},\dots,x_{m,1}),\dots,B_{(m)}(x_{1,n},\dots,x_{m,n})).$$

Let now $\boldsymbol{A}$ and $\boldsymbol{B}$ be aggregation operators. We say that $\boldsymbol{A}$ dominates $\boldsymbol{B}$ if $A_{(n)}$ dominates $B_{(m)}$ for all $n,m\in\mathbb{N}$. We denote it by $\boldsymbol{A}\gg\boldsymbol{B}$.

Remark 3.2. Given that t-norms are associative aggregation operators, $\boldsymbol{A}$ dominates T if and only if, for each $n>1$:

$$A(T(x_1,y_1),\dots,T(x_n,y_n))\ge T(A(x_1,\dots,x_n),A(y_1,\dots,y_n))$$

for all $(x_1,\dots,x_n),(y_1,\dots,y_n)\in[0,1]^n$.

In the context of aggregation of T-subgroups it is necessary to define new forms of domination that are less restrictive in the following sense. If an aggregation dominates T then it preserves the structure of T-subgroup, however there are aggregations that preserve T-subgroups defined over some ambient group in C, but do not dominate T, i.e. domination is a sufficient but not necessary condition for preserving T-subgroups over these ambient groups. The new form of domination reduces the domain over which the inequality in Remark 3.2 must be satisfied to a strict subset of $[0,1]^n\times[0,1]^n$. Hence the set of aggregation operators that dominate a t-norm in these new ways is larger than the set of such functions that dominate the given t-norm in the usual way.

Definition 3.3 *(Different types of domination).* Let A be an n-ary aggregation operator and let T be a t-norm. We define the next three different types of domination as follows:

1. **Type-k domination.** Given $k\in\mathbb{N}$, we say that A type-k dominates T if:

$$A(T(x_1,y_1),\dots,T(x_n,y_n))\ge T(A(x_1,\dots,x_n),A(y_1,\dots,y_n)) \tag{3}$$

for all $(\overline{x},\overline{y})\in S_k\subseteq[0,1]^n\times[0,1]^n$ such that $\overline{x}=(x_1,\dots,x_n),\overline{y}=(y_1,\dots,y_n)$ and:

$$\mathcal{S}_k = \{(\overline{x},\overline{y}) \in [0,1]^n \times [0,1]^n \mid \text{for each } 1 \leq i \leq n \text{ either } \max\{x_i, y_i\} = 1 \text{ or } \min\{x_i,y_i\} \geq (\max\{x_i,y_i\})_T^{(k)}\}.$$

We denote this property by $A \gg_k T$.

2. **Type-k asymmetric domination.** Given $k \in \mathbb{N}$, we say that A type-k asymmetrically dominates T if (3) holds for all $(\overline{x},\overline{y}) \in \mathcal{S}^k \subseteq [0,1]^n \times [0,1]^n$ where:

$$\mathcal{S}^k = \{(\overline{x},\overline{y}) \in [0,1]^n \times [0,1]^n \mid y_i \geq (x_i)_T^{(k)} \text{ for all } 1 \leq i \leq n\}.$$

We denote this property by $A \gg^k T$. We say that A type-∞ asymmetrically dominates T if (3) holds for all:

$$(\overline{x},\overline{y}) \in \mathcal{S}^\infty = \bigcup_{k\in\mathbb{N}} \mathcal{S}^k \subseteq [0,1]^n \times [0,1]^n$$

and we denote it by $A \gg^\infty T$.

3. **Pseudo-p domination.** Given a prime number $p \geq 5$, we say that A pseudo-p dominates T if Inequality (3) holds for all $(\overline{x},\overline{y}) \in \tilde{\mathcal{S}}_p \subseteq [0,1]^n \times [0,1]^n$ where:

$$\tilde{\mathcal{S}}_p = \{ (\overline{x},\overline{y}) \in [0,1]^n \times [0,1]^n \mid \exists k \in \left\{1,\dots,\tfrac{p-3}{2}\right\} \text{ such that for each } i \in \{1,\dots,n\} \text{ either } x_i \geq y_i \geq (x_i)_T^{(k)} \text{ or } y_i \geq x_i \geq (y_i)_T^{(\min\{k',p-k'-2\})} \text{ for the unique } k' \in \{1,\dots,p-1\} \text{ such that } kk' \equiv 1 \mod p \}.$$

We denote the pseudo-p domination of A over T by $A \succcurlyeq_p T$.

We say that an aggregation operator $\boldsymbol{A}$ type-k dominates (type-k asymmetrically dominates/pseudo-p dominates) T if $A_{(n)} \gg_k T$ ($A_{(n)} \gg^k T / A_{(n)} \succcurlyeq_p T$) for all $n \in \mathbb{N}$ and we denote it by $\boldsymbol{A} \gg_k T$ ($\boldsymbol{A} \gg^k T/\ \boldsymbol{A} \succcurlyeq_p T$).

In the following subsections, we provide a detailed description of the characteristics of these newly introduced dominations.

3.1. Type-k domination

Type-k domination was first introduced in [39] to characterize the preservation of T-subgroups on products of some groups whose subgroup lattice is a chain. More precisely, those of prime order. Nevertheless, for the study of the preservation of T-subgroups on sets when G belong to $\mathcal{C}$ (recall that $\mathcal{C}$ is the class of all the groups such that $Lat(G)$ is a chain), we will need to use type-k asymmetric domination and pseudo-p domination. In particular, we will show that type-k asymmetric domination characterizes the aggregation operators that preserve T-subgroups on sets for those cyclic groups of prime power order $p^m > 4$ with $m > 1$. Similarly, it will be proved that pseudo-p domination is a defining characteristic of the aggregation operators that preserve T-subgroups on sets for groups of prime order $p \geq 5$. As previously stated, they restrict in different ways the domain over which the inequality (3) holds.

Remark 3.4. Some properties of type-k domination are:

1. $A \gg_0 T$ if and only if Inequality (3) holds for all $(\overline{x},\overline{y}) \in \mathcal{S}_0 \subseteq [0,1]^n \times [0,1]^n$, where:

$$\mathcal{S}_0 = \{(\overline{x},\overline{y}) \in [0,1]^n \times [0,1]^n \mid \max\{x_i,y_i\} = 1 \text{ for all } 1 \leq i \leq n\}.$$

2. $A \gg_1 T$ if and only if Inequality (3) holds for all $(\overline{x},\overline{y}) \in \mathcal{S}_1 \subseteq [0,1]^n \times [0,1]^n$, where:

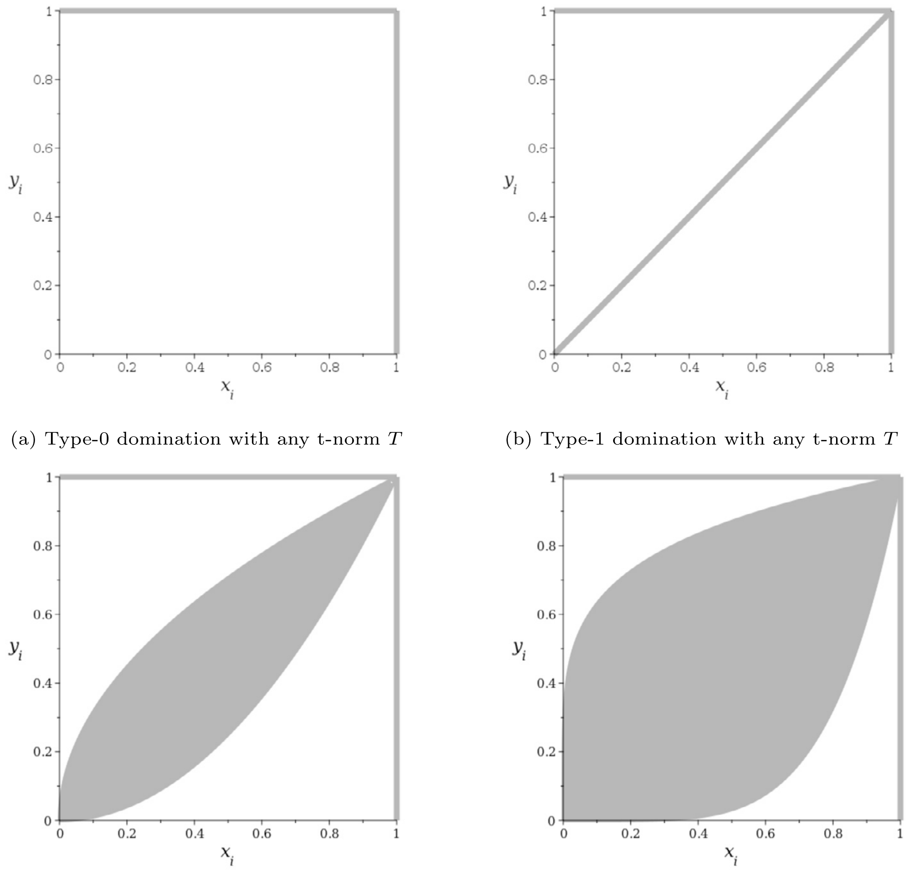
$$\mathcal{S}_1 = \{(\overline{x},\overline{y}) \in [0,1]^n \times [0,1]^n \mid \text{for each } 1 \leq i \leq n \text{ either } \max\{x_i,y_i\} = 1 \text{ or } x_i = y_i\}.$$

3. We can represent in a plot the points that must fulfill Inequality (3) to ensure type-k domination. They will only depend on the t-norm that is being dominated. However, as a consequence of previous items 1 and 2 in this Remark, these representations do not depend on the selected t-norm when $k = 0$ or $k = 1$. We show some examples of this type of domination in Fig. 1. Note that the area of the points where Inequality (3) holds increases with the parameter k.
4. Type-k domination is stronger as we increase the value of k. That is, given $k_1, k_2 \in \mathbb{N}$ where $k_1 \geq k_2$, we have the following chain of implications:

$$A \gg T \Rightarrow A \gg_{k_1} T \Rightarrow A \gg_{k_2} T \Rightarrow A \gg_0 T.$$

This is deduced from the fact that $\mathcal{S}_{k_2} \subseteq \mathcal{S}_{k_1}$.

For more details about this kind of domination we refer to [39].



(a) Type-0 domination with any t-norm T

(b) Type-1 domination with any t-norm T

(c) Type-2 domination using T_P t-norm

(d) Type-5 domination using T_P t-norm

Fig. 1. Graphical illustrations of type-k domination. If $A \gg_k T$, Inequality (3) must hold whenever the point (x_i, y_i) lies in the gray area of the corresponding plot for all $i \in \{1, \dots, n\}$. The four panels illustrate the increase of parameter k in the type-k domination.

3.2. Pseudo-p domination

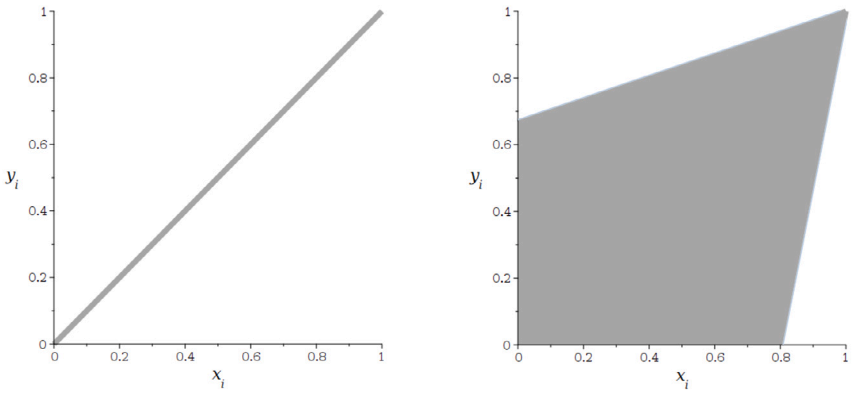
The motivation for the introduction of the term pseudo-p domination originates from restricting the set $[0,1]^n \times [0,1]^n$ with respect to a prime number. The definition of this form of domination may seem at first a little awkward but it is necessary to characterize the preservation of T-subgroups on sets over groups with a prime number of elements. The following remarks clarify the use of this definition.

Remark 3.5.

1. If we set p = 5, it is clear that $A \succcurlyeq_5 T$ if and only if Inequality (3) holds for all $\overline{x} = (x_1, \dots, x_n), \overline{y} = (y_1, \dots, y_n) \in [0,1]^n$ such that $x_i = y_i$ holds for each $i \in \{1, \dots, n\}$ (see Fig. 2a). Comparing this graph with the one associated with type-1 domination (Fig. 1b), it is clear that $A \gg_1 T \Rightarrow A \succcurlyeq_5 T$. Indeed:

$$(\overline{x}, \overline{y}) \in \tilde{S}_5 = \{(\overline{x}, \overline{y}) \in [0,1]^n \times [0,1]^n \mid x_i = y_i \text{ for all } 1 \le i \le n\} \subseteq S_1.$$

2. We can check that pseudo-p domination is well defined. It is important to note that, since in the definition $k \notin \{\frac{p-1}{2}, p-1\}$, then $k' \notin \{p-2, p-1\}$. Consequently $\min\{k', p-k'-2\} \ge 1$ and the inequalities involved in this property always make sense. In other words, the set $\tilde{S}_p$ is always well defined.
3. If $A \succcurlyeq_p T$, since T is commutative, for any $(\overline{x}', \overline{y}') \in \tilde{S}_p$ Inequality (3) also holds if we take $\overline{x} = \overline{y}'$ and $\overline{y} = \overline{x}'$ in Definition 3.3. That is, if we interchange the values of $\overline{x}'$ and $\overline{y}'$.
4. It is easy to prove that $\tilde{S}_p \subseteq S_{\frac{p-3}{2}}$ and hence that $A \gg T \Rightarrow A \gg_{\frac{p-3}{2}} T \Rightarrow A \succcurlyeq_p T$ for each prime $p \ge 5$.
5. As a consequence of the fact that $(x_i)_{T_M}^{(k)} = x_i$ for all $k \in \mathbb{N}$ and given a prime number p, every aggregation operator A pseudo-p dominates the minimum t-norm, i.e. $A \succcurlyeq_p T_M$ for all prime p. Consequently, pseudo-p domination is only interesting when $T \ne T_M$. In [39, Remark 4.7] it is shown that domination over T_M is equivalent to type-0 domination over T_M and thus, to type-k



(a) Pseudo-5 domination with any t-norm T (b) Pseudo-13 domination using T_L t-norm

Fig. 2. Graphical illustrations of pseudo-p domination. If $A \succcurlyeq_p T$, Inequality (3) must hold whenever all the points (x_i, y_i) lie in the gray area of the corresponding plot (a or b) for all $i \in \{1, \dots, n\}$. Panel (a) corresponds to pseudo-5 domination and panel (b) corresponds to pseudo-13 domination using the t-norm T_L.

domination. These two situations reinforce the idea that pseudo-p domination is a weaker property than type-k domination. A weak form that will prove to be very useful in the following developments.

3.3. Type-k asymmetric domination

This new modification of the definition of domination will prove to be relevant in the present work. The reason for calling this property type-k asymmetric domination is justified in the next remark. As in the previous cases, there are some interesting features of type-k asymmetric domination that are worth mentioning.

Remark 3.6.

1. Note that $(x_i)_T^{(k)} \geq (x_i)_T^{(k+1)} \geq 0$ for all $k \geq 1$. Therefore, $\mathcal{S}^k \subseteq \mathcal{S}^{k+1} \subseteq \mathcal{S}^\infty$ and $A \gg^\infty T \Rightarrow A \gg^{k+1} T \Rightarrow A \gg^k T$. Thus, given $k_1, k_2 \in \mathbb{N}$ with $k_1 \geq k_2$ we have that $A \gg T \Rightarrow A \gg^\infty T \Rightarrow A \gg^{k_1} T \Rightarrow A \gg^{k_2} T \Rightarrow A \gg^1 T$.
2. If some points $\overline{x} = (x_1, \dots, x_n), \overline{y} = (y_1, \dots, y_n) \in [0,1]^n$ are such that $(\overline{x}, \overline{y}) \in \tilde{\mathcal{S}}_p$ for a prime p, then it is clear that $y_i \geq (x_i)_T^{\left(\frac{p-3}{2}\right)}$ for all $i \in \{1, \dots, n\}$. Hence $\tilde{\mathcal{S}}_p \subseteq \mathcal{S}^{\frac{p-3}{2}}$ and $A \gg^{\frac{p-3}{2}} T \Rightarrow A \succcurlyeq_p T$.
3. The set of points $\{(x_i, y_i) \mid i \in \{1, \dots, n\}\}$ where Inequality (3) must hold in order to ensure that A type-k asymmetrically dominates T depends on the used t-norm T. Thus, for each T we can represent the subset of $[0,1]^2$ where (x_i, y_i) must lie for all $i \in \{1, \dots, n\}$. That is, the points fulfilling the inequality $y_i \geq (x_i)_T^{(k)}$. We can find some examples of these representations in Fig. 3. Note that we use the word "asymmetric" because the plot of these subsets is not symmetrical with respect to the main diagonal as it is the case with type-k domination (compare Figs. 1d and 3c).
4. As it occurs in the case of pseudo-p domination, $\boldsymbol{A} \gg^k T_M$ for all $k \in \mathbb{N}$ and for all aggregation operator $\boldsymbol{A}$. Therefore, the type-k asymmetric domination only makes sense when we use a t-norm different from the minimum.

To show that type-k asymmetric domination and type-k domination are indeed different properties we present the next example.

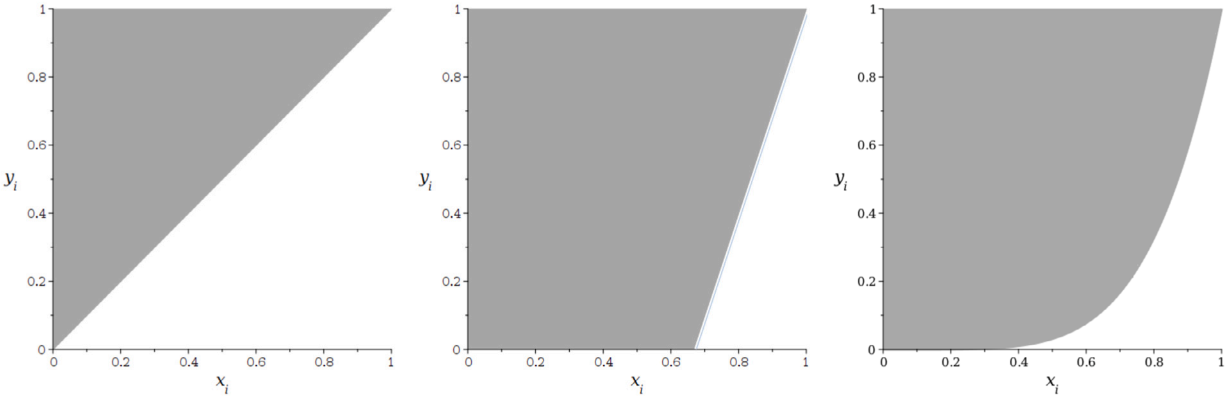
Example 3.7. Let us analyze some examples that highlight the existing differences between type-k asymmetric domination, domination and type-k domination. More precisely, they are counterexamples that show the non-equivalence between type-k asymmetric domination and the other two types of domination. Let us define the next t-norms:

$$T_1(x,y) = \begin{cases} 0 & \text{if } (x,y) \in]0, 0.5[^2, \\ \min\{x,y\} & \text{otherwise,} \end{cases}$$

$$T_2(x,y) = \begin{cases} 0 & \text{if } (x,y) \in]0, 1[^2 \setminus [0.5, 1[^2, \\ \min\{x,y\} & \text{otherwise,} \end{cases}$$

and the aggregation operator $\boldsymbol{A}_1$ such that, for $n \in \mathbb{N}$:

$$A_1(x_1, \dots, x_n) = \begin{cases} 1 & \text{if } x_1 \in [0.5, 1], \\ x_1 & \text{otherwise.} \end{cases}$$



(a) Type-1 asymmetric domination over any t-norm T

(b) Type-3 asymmetric domination using T_L t-norm

(c) Type-5 asymmetric domination using T_P t-norm

Fig. 3. Illustrations of type-k asymmetric domination. The shaded areas of the corresponding plots display the possible values of (x_i, y_i) that must satisfy Inequality (3) to ensure that $A \gg^k T$.

1. It is easy to check that $T_1 \geq T_2$ and consequently that $T_2 \not\gg T_1$. However, let us show that $T_2 \gg^k T_1$ for all $k \in \mathbb{N}$. That is, given $(x_1, \dots, x_n), (y_1, \dots, y_n) \in [0,1]^n$ and $k \in \mathbb{N}$ with $y_i \geq (x_i)_{T_1}^{(k)}$ for each $i \in \{1, \dots, n\}$, the following inequality holds:

$$T_2(T_1(x_1, y_1), \dots, T_1(x_n, y_n)) \geq T_1(T_2(x_1, \dots, x_n), T_2(y_1, \dots, y_n)).$$

Since

$$(x_i)_{T_1}^{(k)} = \begin{cases} 0 & \text{if } x_i \in [0, 0.5[, \\ x_i & \text{otherwise,} \end{cases}$$

the statement $y_i \geq (x_i)_{T_1}^{(k)}$ allows only 3 possibilities:
(a) $0.5 > y_i \geq 0$ and $0.5 > x_i \geq 0$,
(b) $y_i \geq 0.5 > x_i \geq 0$,
(c) $1 \geq y_i \geq x_i \geq 0.5$.
In both cases (b) and (c) we have that $T_1(x_i, y_i) = \min\{x_i, y_i\} = x_i$. Therefore if we suppose that case (a) does not happen for any $i \in \{1, \dots, n\}$ we conclude that:

$$\begin{aligned} T_2(T_1(x_1, y_1), \dots, T_1(x_n, y_n)) &= T_2(x_1, \dots, x_n) \\ &\geq T_1(T_2(x_1, \dots, x_n), T_2(y_1, \dots, y_n)). \end{aligned}$$

Conversely, if there exists $i_0 \in \{1, \dots, n\}$ such that case (a) occurs, then $T_1(x_{i_0}, y_{i_0}) = 0$. In addition $0.5 > y_{i_0} \geq T_2(y_1, \dots, y_n)$ and $0.5 > x_{i_0} \geq T_2(x_1, \dots, x_n)$. Hence:

$$T_2(T_1(x_1, y_1), \dots, T_1(x_n, y_n)) = 0 = T_1(T_2(x_1, \dots, x_n), T_2(y_1, \dots, y_n)).$$

In any case, $T_2 \gg^k T_1$.
2. Using the same operators as in point 1, we can check that type-k domination and type-k asymmetric domination are also not equivalent since:

$$\begin{aligned} T_2(T_1(1, 0.25), T_1(0.75, 1)) = T_2(0.25, 0.75) = 0 < 0.25 &= \min\{0.75, 0.25\} \\ &= T_1(0.75, 0.25) = T_1(T_2(1, 0.75), T_2(0.25, 1)). \end{aligned}$$

As a consequence, and keeping in mind that $\max\{1, 0.25\} = \max\{0.75, 1\} = 1$, we conclude that $T_2 \not\gg_k T_1$.
3. Finally, we can prove that type-k domination does not imply type-k asymmetric domination. In [39, Example 4.19] it was shown that $\boldsymbol{A}_1 \gg_k T_2$ for all $k \in \mathbb{N}$. Just the opposite situation occurs when we talk about the relation for property $\gg^k$. Given $x_1 = 0.25$, $y_1 = 0.75$ and $1 \geq y_2 \geq x_2 \geq 0$. Clearly, for all $k \geq 2$ and for each $i \in \{1, 2\}$:

$$y_i \geq (x_i)_{T_2}^{(k)} = \begin{cases} 0 & \text{if } x_i \in [0, 0.5[, \\ x_i & \text{otherwise.} \end{cases}$$

Moreover:

$$A_1(T_2(x_1, y_1), T_2(x_2, y_2)) = A_1(T_2(0.25, 0.75), T_2(x_2, y_2)) = A_1(0, T_2(x_2, y_2))$$

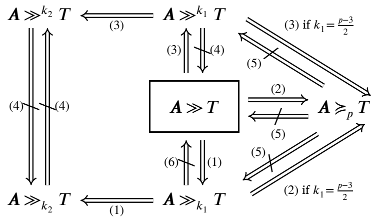


Fig. 4. Connections between domination, type-k domination, type-k asymmetric domination and pseudo-p domination. Note that we use arbitrary $k_1, k_2 \in \mathbb{N}$ such that $k_1 \geq k_2$. The results that show each implication are: (1) Remark 3.4, (2) Remark 3.5, (3) Remark 3.6, (4) Example 3.7, (5) Remark 3.8, (6) Example 4.19 in [39].

$$= 0 < 0.25 = T_2(0.25, 1) = T_2(A_1(0.25, x_2), A_1(0.75, y_2)).$$

Thus, $\boldsymbol{A}_1 \not\gg^k T_2$.

Additional statements that relate the different types of domination can be deduced.

Remark 3.8.

1. Given a prime p, note that in Example 3.7 we have that $T_2 \gg^{\frac{p-3}{2}} T_1$ but $T_2 \not\gg_{\frac{p-3}{2}} T_1$. Therefore, from Remark 3.6 we know that $T_2 \geqslant_p T_1$. This means that $\geqslant_p$ and $\gg_{\frac{p-3}{2}}$ are not equivalent properties.
2. Moreover, since $T_2 \not\gg T_1$, we get that $\geqslant_p$ and $\gg$ are also different properties.
3. Again, in Example 3.7 we have that $\boldsymbol{A}_1 \not\gg^{\frac{p-3}{2}} T_2$ and $\boldsymbol{A}_1 \gg_{\frac{p-3}{2}} T_2$. Hence $\boldsymbol{A}_1 \geqslant_p T_2$ (see Remark 3.5) and therefore type-k asymmetric domination and pseudo-p domination are not equivalent.

The diagram of Fig. 4 summarizes the currently known relationships between the different forms of domination. In the following section we will show some results to facilitate the use of these new relations.

3.4. Aggregation, t-norms and relaxed dominations

The study of the relationship between aggregation operators under domination has been developed in a large number of cases. Although, there exists no general theory to obtain aggregation operators dominating an arbitrary aggregation operator, an intense search on conditions for domination in general and in specific families of aggregations such as several types of t-norms have been undertaken (to mention some articles [22,28,29,32–37,40]).

In [33], the authors obtained interesting results about the dominance relation that can be directly applied to our new types of domination, since they are based on a restriction on the set of points $[0,1]^n \times [0,1]^n$ where domination is defined. The next result is inspired by Proposition 2.8 in [33].

Proposition 3.9. *Let $\boldsymbol{A}$ be an associative aggregation operator and T a t-norm. Then, the following statements hold:*

(i) *For a given $k \in \mathbb{N}$, if $A_{(2)} \gg^k T$ then $\boldsymbol{A} \gg^k T$.*
(ii) *For a given prime number $p \geq 5$, if $A_{(2)} \geqslant_p T$ then $\boldsymbol{A} \geqslant_p T$.*

Proof. To prove (i) we need first to show that, for an arbitrary $n \in \mathbb{N}$, if $A_{(n)} \gg^k T$ then:

$$A\left((x_1)_T^{(j)}, \dots, (x_n)_T^{(j)}\right) \geq A(x_1, \dots, x_n)_T^{(j)} \tag{4}$$

for all $(x_1, \dots, x_n) \in [0,1]^n$ and $1 \leq j \leq k$. First note that:

$$A\left((x_1)_T^{(1)}, \dots, (x_n)_T^{(1)}\right) = A(x_1, \dots, x_n) = A(x_1, \dots, x_n)_T^{(1)}.$$

Let us suppose then that $1 < k$ and we will check that, while $j \leq k$, if equation (4) holds for $j-1$, it also holds for j. That is, we will suppose that:

$$A\left((x_1)_T^{(j-1)}, \dots, (x_n)_T^{(j-1)}\right) \geq A(x_1, \dots, x_n)_T^{(j-1)}.$$

In this situation:

$$A\left((x_1)_T^{(j)}, \dots, (x_n)_T^{(j)}\right) = A\left(T(x_1, (x_1)_T^{(j-1)}), \dots, T(x_n, (x_n)_T^{(j-1)})\right)$$

$$\geq T\left(A(x_1,\dots,x_n), A((x_1)_T^{(j-1)},\dots,(x_n)_T^{(j-1)})\right) \tag{5}$$
$$\geq T\left(A(x_1,\dots,x_n), A(x_1,\dots,x_n)_T^{(j-1)}\right)$$
$$= A(x_1,\dots,x_n)_T^{(j)}$$

where the first inequality comes from $A_{(n)} \gg^k T$ and $(x_i)_T^{(j-1)} \geq (x_i)_T^{(k)}$. The second one is a consequence of the hypothesis and the monotonicity of T.

With this preliminary property we can show by induction over n that $A_{(n)} \gg^k T$ for all $n > 2$. The first step is given by the hypothesis $A_{(2)} \gg^k T$. Let us assume that $A_{(n-1)} \gg^k T$ and try to find out if $A_{(n)} \gg^k T$. For each $(\overline{x}, \overline{y}) \in \mathcal{S}^k \subseteq [0,1]^n \times [0,1]^n$ with $y_i \geq (x_i)_T^{(k)}$ for all $i \in \{1,\dots,n\}$ we have:

$$\begin{aligned}\mathbf{A}(T(x_1,y_1),\dots,T(x_n,y_n)) &= \mathbf{A}(\mathbf{A}(T(x_1,y_1),\dots,T(x_{n-1},y_{n-1})),T(x_n,y_n))\\ &\geq \mathbf{A}(T(\mathbf{A}(x_1,\dots,x_{n-1}),\mathbf{A}(y_1,\dots,y_{n-1})),T(x_n,y_n))\\ &\geq T(\mathbf{A}(\mathbf{A}(x_1,\dots,x_{n-1}),x_n),\mathbf{A}(\mathbf{A}(y_1,\dots,y_{n-1}),y_n))\\ &= T(\mathbf{A}(x_1,\dots,x_n),\mathbf{A}(y_1,\dots,y_n)).\end{aligned} \tag{6}$$

The two identities in the above expression are a consequence of the associativity of $\mathbf{A}$. The first inequality is due to $A_{(n-1)} \gg^k T$. Moreover, the last inequality can be deduced from $A_{(2)} \gg^k T$ since the inequality (4) shown at the beginning of the proof ensures that:

$$A(y_1,\dots,y_{n-1}) \geq A\left((x_1)_T^{(k)},\dots,(x_{n-1})_T^{(k)}\right) \geq A(x_1,\dots,x_{n-1})_T^{(k)}$$

because $A_{(n-1)} \gg^k T$ and $y_i \geq (x_i)_T^{(k)}$ for all $i \in \{1,\dots,n\}$.

To prove statement (ii) we can show by a similar reasoning as in (i) that if $A_{(n)} \succcurlyeq_p T$ then (4) holds for all $(x_1,\dots,x_n) \in [0,1]^n$ and $1 \leq j \leq \max\{k,r\}$ where $r = \min\{k', p-k'-2\}$. The only important difference in this case is that $x_i \geq (x_i)_T^{(j-1)} \geq (x_i)_T^{(\max\{k,r\})}$ for all $i \in \{1,\dots,n\}$ and thus, either $(\overline{x},\overline{x}')$ or $(\overline{x}',\overline{x})$ belong to $\tilde{\mathcal{S}}_p$ when $\overline{x} = (x_1,\dots,x_n)$ and $\overline{x}' = ((x_1)_T^{(j-1)},\dots,(x_n)_T^{(j-1)})$. Therefore, by Remark 3.5, expression (5) also holds in this case.

Induction is going to be used again to prove this case. Let us suppose that $A_{(n-1)} \succcurlyeq_p T$ and let us set $(\overline{x},\overline{y}) \in \tilde{\mathcal{S}}_p \subseteq [0,1]^n \times [0,1]^n$ with either $x_i \geq y_i \geq (x_i)_T^{(k)}$ or $y_i \geq x_i \geq (y_i)_T^{(r)}$ for all $i \in \{1,\dots,n\}$. This is equivalent to say that $y_i \geq (x_i)_T^{(k)}$ and $x_i \geq (y_i)_T^{(r)}$ for all $i \in \{1,\dots,n\}$ by recalling that $x_i \geq (x_i)_T^{(l)}$ and $y_i \geq (y_i)_T^{(l)}$ for all $l \in \mathbb{N}$. Therefore, from inequality (4) and due to the induction hypothesis:

$$A(y_1,\dots,y_{n-1}) \geq A((x_1)_T^{(k)},\dots,(x_{n-1})_T^{(k)}) \geq A(x_1,\dots,x_{n-1})_T^{(k)} \text{ and}$$
$$A(x_1,\dots,x_{n-1}) \geq A((y_1)_T^{(r)},\dots,(y_{n-1})_T^{(r)}) \geq A(y_1,\dots,y_{n-1})_T^{(r)}.$$

Consequently, the identities and inequalities in (6) are also true in this case. □

Remark 3.10. As a consequence of Proposition 3.9 to prove that two t-norms are related by $\gg^k$ or $\succcurlyeq_p$ we only need to check Definition 3.3 for $n = 2$.

At this point, we will provide some interesting properties of t-norms regarding different types of domination. The subsequent proposition constitutes an extension of Proposition 7 in [32], which addresses the domination between conjunctors. Its proof is analogous to the one given in that result.

Proposition 3.11. *Let T_1 and T_2 be a pair of t-norms, $k \geq 1$ a natural number and $p \geq 5$ a prime number. If $T_1 \gg_k T_2$, $T_1 \gg^k T_2$ or $T_1 \succcurlyeq_p T_2$, then the operator T_2 is closed in the set of idempotent elements of T_1 given by:*

$$I(T_1) = \{x \in [0,1] \mid T_1(x,x) = x\}.$$

i.e. $T_2(x,y) \in I(T_1)$ for all $x, y \in I(T_1)$.

This last result is useful to prove that two t-norms are not related by some types of domination as it can be checked in the proof of Theorem 3.14.

To motivate the following result recall that given two arbitrary t-norms T_1 and T_2, it is well known that if $T_1 \gg T_2$, then $T_1 \geq T_2$. The same occurs when $T_1 \gg_k T_2$ as a consequence of the following lemma.

Lemma 3.12. *Given $k \in \mathbb{N}$. If T_1 and T_2 are two t-norms such that $T_1 \gg_k T_2$, then $T_1 \geq T_2$.*

Proof. Let $x_2 = 1$, $y_1 = 1$ and $x_1, y_2 \in [0,1]$ be real numbers. Hence, $\max\{x_i, y_i\} = 1$ for all $i \in \{1,2\}$ and (3) in Definition 3.3 holds for these points. Consequently:

$$T_1(x_1,y_2) = T_1(T_2(x_1,y_1),T_2(x_2,y_2)) \geq T_2(T_1(x_1,x_2),T_1(y_1,y_2)) = T_2(x_1,y_2)$$

and $T_1 \geq T_2$. □

However, this situation is not true in the case of type-k asymmetric domination and pseudo-p domination as we will see in Remark 3.15 as a consequence of the following theorems. Note that Saminger, Mesiar and De Meyer proved in [32] that the only domination relations in the families of Dubois-Prade and Mayor-Torrens t-norms were trivial (self dominance and T_M dominating every other t-norm in the family to be precise). We will show that the relationship between members of these families is much richer when we consider the $\gg^k$ and $\succcurlyeq_p$ dominations.

Theorem 3.13. *Let $\{T_\lambda^{\mathbf{DP}}\}_{\lambda\in[0,1]}$ be the Dubois-Prade family of t-norms given for each $\lambda \in [0,1]$ by:*

$$T_\lambda^{\mathbf{DP}}(x,y) = \begin{cases} \frac{xy}{\lambda} & \text{if } \lambda \neq 0 \text{ and } (x,y) \in [0,\lambda]^2, \\ \min\{x,y\} & \text{otherwise,} \end{cases}$$

where $T_0^{\mathbf{DP}} = T_M$. Then, for a given $k \in \mathbb{N}$ we have that $T_{\lambda_1}^{\mathbf{DP}} \gg^k T_{\lambda_2}^{\mathbf{DP}}$ if and only if $\lambda_1 \in \{0\} \cup [\lambda_2, 1]$. Moreover, given a prime number $p \geq 5$ we have $T_{\lambda_1}^{\mathbf{DP}} \succcurlyeq_p T_{\lambda_2}^{\mathbf{DP}}$ if and only if $\lambda_1 \in \{0\} \cup [\lambda_2, 1]$.

Proof. Let us check that if $\lambda_1 \in \{0\} \cup [\lambda_2, 1]$, then $T_{\lambda_1}^{\mathbf{DP}} \gg^k T_{\lambda_2}^{\mathbf{DP}}$ for each $k \in \mathbb{N}$. Since $T_0^{\mathbf{DP}} = T_M$ and $T_M \gg T$ for all t-norm T, we have that $T_{\lambda_1}^{\mathbf{DP}} \gg^k T_{\lambda_2}^{\mathbf{DP}}$, whenever $\lambda_1 = 0$. Along the proof, we will use for simplicity $T_1 = T_{\lambda_1}^{\mathbf{DP}}$ and $T_2 = T_{\lambda_2}^{\mathbf{DP}}$. If $\lambda_2 = 0$, then for $x_i \geq y_i = (y_i)_{T_M}^{(k)}$ for all $i \in \{1,2\}$, clearly:

$$T_1(T_2(x_1,y_1),T_2(x_2,y_2)) = T_1(y_1,y_2) \geq T_2(T_1(x_1,x_2),T_1(y_1,y_2)).$$

Moreover, if $\lambda_1 = \lambda_2$ the proof is straightforward.

Let us suppose then $\lambda_1 > \lambda_2 > 0$. Given $(\overline{x},\overline{y}) \in S^k$ such that for each $1 \leq i \leq 2$:

$$x_i \geq (y_i)_{T_2}^{(k)} = \begin{cases} y_i & \text{if } y_i > \lambda_2, \\ \frac{(y_i)^k}{\lambda_2^{k-1}} & \text{if } y_i \leq \lambda_2, \end{cases} \tag{7}$$

we need to discuss several cases:

1. If $\max\{x_i, y_i\} > \lambda_2$ for all $i \in \{1,2\}$, then we can deduce that $x_i \geq y_i$. Note that if $y_i > \lambda_2$, by Equation (7) we have $x_i \geq y_i$ and if $\lambda_2 \geq y_i$ then $x_i > \lambda_2 \geq y_i$. Thus:

$$\begin{aligned} T_1(T_2(x_1,y_1),T_2(x_2,y_2)) &= T_1(\min\{x_1,y_1\},\min\{x_2,y_2\}) = T_1(y_1,y_2) \\ &\geq T_2(T_1(x_1,x_2),T_1(y_1,y_2)). \end{aligned}$$

2. If $\max\{x_i, y_i\} \leq \lambda_2 < \lambda_1$ for all $i \in \{1,2\}$:

$$\begin{aligned} T_1(T_2(x_1,y_1),T_2(x_2,y_2)) &= \frac{x_1x_2y_1y_2}{\lambda_1\lambda_2^2} \\ &\geq \frac{x_1y_1x_2y_2}{\lambda_1^2\lambda_2} = T_2(T_1(x_1,x_2),T_1(y_1,y_2)). \end{aligned}$$

3. If $\max\{x_1, y_1\} \leq \lambda_2 < \lambda_1$ and $\max\{x_2, y_2\} > \lambda_2$ then $x_2 \geq y_2$ and we have two new cases:
   3a. Whenever $y_2 > \lambda_1$:

$$\begin{aligned} T_1(T_2(x_1,y_1),T_2(x_2,y_2)) &= T_1\left(T_2(x_1,y_1),\min\{x_2,y_2\}\right) \\ &= T_1(T_2(x_1,y_1),y_2) = T_2(x_1,y_1) \geq T_2(T_1(x_1,x_2),T_1(y_1,y_2)) \end{aligned}$$

   since $T_2(x_1,y_1) \leq \min\{x_1,y_1\} < \lambda_1 < y_2$.
   3b. If $y_2 \leq \lambda_1$:

$$\begin{aligned} T_1(T_2(x_1,y_1),T_2(x_2,y_2)) &= T_1\left(T_2(x_1,y_1),\min\{x_2,y_2\}\right) \\ &= T_1\left(\frac{x_1y_1}{\lambda_2},y_2\right) = \frac{x_1y_1y_2}{\lambda_1\lambda_2} = T_2\left(x_1,\frac{y_1y_2}{\lambda_1}\right) \end{aligned}$$

$$= T_2(T_1(x_1,x_2),T_1(y_1,y_2)).$$

4. The case where $\max\{x_1,y_1\} > \lambda_2$ and $\max\{x_2,y_2\} \le \lambda_2 < \lambda_1$ is analogous to case 3.

As a direct consequence of the previous discussion and by Remark 3.6 we will have that $T^{\mathbf{DP}}_{\lambda_1} \gg_p T^{\mathbf{DP}}_{\lambda_2}$ for each prime $p \ge 5$ and for all $\lambda_1 \in \{0\} \cup [\lambda_2, 1]$. Now, we will show that no other values of λ_1 and λ_2 lead to neither type-k asymmetric domination nor pseudo-p domination. Let us suppose that $0 < \lambda_1 < \lambda_2$. In this case, if we take $0 < x < \lambda_1$:

$$T_1(T_2(x,x),T_2(x,x)) = \frac{x^4}{\lambda_2^2\lambda_1} < \frac{x^4}{\lambda_1^2\lambda_2} = T_2(T_1(x,x),T_1(x,x)).$$

Since, given $\overline{x} = (x,x)$, $(\overline{x},\overline{x}) \in \mathcal{S}^k$ for all $k \in \mathbb{N}$ and $(\overline{x},\overline{x}) \in \tilde{\mathcal{S}}_p$ for all prime $p \ge 5$, then $T_1 \not\gg^k T_2$ and $T_1 \not\gg_p T_2$. □

Theorem 3.14. *Let $\{T^{\mathbf{MT}}_\lambda\}_{\lambda\in[0,1]}$ be the Mayor-Torrens family of t-norms given for each $\lambda \in [0,1]$ by:*

$$T^{\mathbf{MT}}_\lambda(x,y) = \begin{cases} \max\{x+y-\lambda, 0\} & \text{if } (x,y) \in [0,\lambda]^2, \\ \min\{x,y\} & \text{otherwise,} \end{cases}$$

where $T^{\mathbf{MT}}_0 = T_M$. Then, for a given $k \in \mathbb{N}$ we have that $T^{\mathbf{MT}}_{\lambda_1} \gg^k T^{\mathbf{MT}}_{\lambda_2}$ if and only if $\lambda_1 \in \{0\} \cup [\lambda_2, 1]$. Moreover, given a prime number p, we have $T^{\mathbf{MT}}_{\lambda_1} \gg_p T^{\mathbf{MT}}_{\lambda_2}$ if and only if $\lambda_1 \in \{0\} \cup [\lambda_2, 1]$.

Proof. Let us check that $T^{\mathbf{MT}}_{\lambda_1} \gg^k T^{\mathbf{MT}}_{\lambda_2}$ for all $\lambda_1 \in \{0\} \cup [\lambda_2, 1]$. We will denote for simplicity $T^{\mathbf{MT}}_{\lambda_1} = T_1$ and $T^{\mathbf{MT}}_{\lambda_2} = T_2$. Note that the proof for $\lambda_2 = 0$ and $\lambda_1 \in \{0, \lambda_2\}$ is the same as the one given for Dubois-Prade t-norms in Theorem 3.13. Hence, let us suppose that $\lambda_1 > \lambda_2 > 0$. Given $(\overline{x},\overline{y}) \in \mathcal{S}^k$ such that for each $1 \le i \le 2$, we have:

$$x_i \ge (y_i)^{(k)}_{T_2} = \begin{cases} y_i & \text{if } y_i > \lambda_2, \\ \max\{ky_i - (k-1)\lambda_2, 0\} & \text{if } y_i \le \lambda_2. \end{cases} \tag{8}$$

We will consider the same cases as in the proof of Theorem 3.13.

1. It is analogous to case 1 in Theorem 3.13.
2. If $\max\{x_i,y_i\} \le \lambda_2 < \lambda_1$ for all $i \in \{1,2\}$:

$$\begin{aligned} &T_1(T_2(x_1,y_1),T_2(x_2,y_2)) = \\ &\quad= \max\{\max\{x_1+y_1-\lambda_2,0\} + \max\{x_2+y_2-\lambda_2,0\} - \lambda_1, 0\} \\ &\quad= \max\{x_1+y_1+x_2+y_2-2\lambda_2-\lambda_1, x_1+y_1-\lambda_2-\lambda_1, \\ &\qquad\qquad x_2+y_2-\lambda_2-\lambda_1, -\lambda_1, 0\} \\ &\quad= \max\{x_1+y_1+x_2+y_2-2\lambda_2-\lambda_1, 0\} \\ &\quad\ge \max\{x_1+y_1+x_2+y_2-2\lambda_1-\lambda_2, 0\} \\ &\quad= T_2(T_1(x_1,x_2),T_1(y_1,y_2)). \end{aligned}$$

3. If $\max\{x_1,y_1\} \le \lambda_2 < \lambda_1$ and $\max\{x_2,y_2\} > \lambda_2$, then $x_2 \ge y_2$.
   3a. It is analogous to case 3a in Theorem 3.13.
   3b. If $y_2 \le \lambda_1$:

$$\begin{aligned} T_1(T_2(x_1,y_1),T_2(x_2,y_2)) &= T_1\left(T_2(x_1,y_1), \min\{x_2,y_2\}\right) = T_1(T_2(x_1,y_1),y_2) \\ &= \max\{\max\{x_1+y_1-\lambda_2,0\} + y_2 - \lambda_1, 0\} \\ &= \max\{x_1+y_1+y_2-\lambda_2-\lambda_1, 0\} \\ &= \max\{x_1 + \max\{y_1+y_2-\lambda_1,0\} - \lambda_2, 0\} \\ &\ge T_2(T_1(x_1,x_2),T_1(y_1,y_2)). \end{aligned}$$

4. The case where $\max\{x_1,y_1\} > \lambda_2$ and $\max\{x_2,y_2\} \le \lambda_2 < \lambda_1$ is analogous to case 3.

This proves that $T^{\mathbf{MT}}_{\lambda_1} \gg^k T^{\mathbf{MT}}_{\lambda_2}$ for all $\lambda_1 \in \{0\} \cup [\lambda_2, 1]$. Additionally, by Remark 3.6, $T^{\mathbf{MT}}_{\lambda_1} \gg_p T^{\mathbf{MT}}_{\lambda_2}$ for all $\lambda_1 \in \{0\} \cup [\lambda_2, 1]$ and for all prime $p \ge 5$. It only remains to show that there is not any other situation in this family where type-k asymmetric domination or pseudo-p domination holds. However, if we suppose $0 < \lambda_1 < \lambda_2$, the discussion provided in [32] Section 5.2.2 to show that $T_1 \not\gg T_2$ is also valid to show that $T_1 \not\gg^k T_2$ and $T_1 \not\gg_p T_2$ by means of Proposition 3.11. □

Remark 3.15. Given a fixed $k \in \mathbb{N}$ and a fixed prime number $p \geq 5$, both families of t-norms are decreasing with respect to the parameter λ, if $\lambda_1 > \lambda_2 > 0$ then $T^{\mathbf{DP}}_{\lambda_2} \geq T^{\mathbf{DP}}_{\lambda_1}$ and $T^{\mathbf{MT}}_{\lambda_2} \geq T^{\mathbf{MT}}_{\lambda_1}$. Thus, by Lemma 3.12, $T^{\mathbf{DP}}_{\lambda_1} \not\gg_k T^{\mathbf{DP}}_{\lambda_2}$, $T^{\mathbf{DP}}_{\lambda_1} \not\gg T^{\mathbf{DP}}_{\lambda_2}$, $T^{\mathbf{MT}}_{\lambda_1} \not\gg_k T^{\mathbf{MT}}_{\lambda_2}$ and $T^{\mathbf{MT}}_{\lambda_1} \not\gg T^{\mathbf{MT}}_{\lambda_2}$. However, for the same values $\lambda_1 > \lambda_2 > 0$ and using Theorems 3.13 and 3.14 we have $T^{\mathbf{DP}}_{\lambda_1} \gg^k T^{\mathbf{DP}}_{\lambda_2}$, $T^{\mathbf{MT}}_{\lambda_1} \gg^k T^{\mathbf{MT}}_{\lambda_2}$, $T^{\mathbf{DP}}_{\lambda_1} \gg_p T^{\mathbf{DP}}_{\lambda_2}$ and $T^{\mathbf{MT}}_{\lambda_1} \gg_p T^{\mathbf{MT}}_{\lambda_2}$. The existence of this infinite number of operators related by pseudo-p domination and type-k asymmetric domination but not related neither by usual domination nor type-k domination will be important in Section 6 to emphasize the difference between aggregation of T-subgroups on products and aggregation of T-subgroups on sets for those groups whose subgroup lattice is a chain.

4. Aggregation of T-subgroups: general framework

As mentioned in the introduction, we aim at studying the aggregation on sets of T-subgroups of groups whose subgroup lattice is a chain, i.e. of groups in C. The other cases regarding aggregation of T-subgroups have been previously studied in [3,39]. First, we recall the main results of these previous works and get an overview of the current studies regarding the aggregation of T-subgroups.

Due to the following proposition, we can prove whether or not $\boldsymbol{A}$ preserves T-subgroups of G by simply checking whether or not an aggregation operator preserves the fuzzy subgroup axiom G3.

Proposition 4.1 *([3]). Let G be a non-trivial group, $\boldsymbol{A} : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ an aggregation operator, and $\mu_1,\ldots,\mu_n$ T-subgroups, then the fuzzy subgroup axioms G1 and G2 are satisfied by $A\circ\boldsymbol{\mu}$ and $A\circ\tilde{\boldsymbol{\mu}}$.*

The next result connects the preservation of T-subgroups on sets and on products.

Proposition 4.2 *([3]). Let G be a non trivial group, T a t-norm, and $\boldsymbol{A}$ an aggregation function. If $\boldsymbol{A}$ preserves the structure of T-subgroup on products then $\boldsymbol{A}$ also preserves this structure on sets.*

Given an arbitrary group G, the preservation of T-subgroups on products always implies the preservation of T-subgroups on sets. We will show later that the converse implication is not true in general.

The domination relation between two aggregation operators is key in the context of aggregation of T-subgroups. It will be a sufficient condition to ensure the preservation of this structure on both, products and sets.

Theorem 4.3 *([3]). Let G be a group and T a t-norm. If $\boldsymbol{A}$ is an aggregation operator that dominates T, then $\boldsymbol{A}$ preserves the structure of T-subgroup on products and hence on sets.*

Therefore, the conditions that an aggregation operator must satisfy in order to preserve T-subgroups are equally or less restrictive than the conditions for domination. For instance, when $G \notin C$ the dominance relation is the only property we need to characterize the preservation of T-subgroups on both sets and products.

Theorem 4.4 *([3]). Let be a group $G \notin C$, T a t-norm and $\boldsymbol{A} : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ an aggregation operator. Then, the following statements are equivalent:*

(i) *$\boldsymbol{A}$ dominates T.*
(ii) *$\boldsymbol{A}$ preserves the structure of T-subgroup on sets.*
(iii) *$\boldsymbol{A}$ preserves the structure of T-subgroup on products.*

As we will see, the situation when $G \in C$ is much more complicated. The only finite groups where $Lat(G)$ is a chain are all the cyclic groups with p^m elements being p a prime number and $m \in \mathbb{N}$. These groups are essential in different branches of Group Theory and they are useful in applications such as cryptography. Moreover, the only infinite groups in C are the Prüfer p-groups $\mathbb{Z}(p^\infty)$ for a prime p. A Prüfer p-group can be defined as the set of the p^mth complex roots of unity for all $m \in \mathbb{N}$ with the complex multiplication as the operation. However, it admits different definitions. It is also important to recall that $Lat(\mathbb{Z}(p^\infty))$ is a chain since each of the proper subgroups of $\mathbb{Z}(p^\infty)$ are isomorphic to $\mathbb{Z}_{p^m}$ for some $m \in \mathbb{N}$ (see [18] for more details). One full article was devoted only to the study of the preservation of T-subgroups on products in this context (see [39]). By using the correct notation, the main results in [39] concerning these groups can be combined in the next theorem.

Theorem 4.5. *Let $G = \langle g \rangle$ be a group such that $|G| = r \in \{p \mid p \text{ is prime}\} \cup \{4\}$, $\boldsymbol{A}$ an aggregation operator and T a t-norm. The following assertions are equivalent:*

(i) *$\boldsymbol{A}$ preserves T-subgroups of G on products.*
(ii) $\boldsymbol{A} \gg_{\left[\frac{r}{2}\right]-1} T$

For the rest of the cases, the domination becomes relevant again since it characterizes the preservation of T-subgroups on products as we can see in the following theorems.

Theorem 4.6 *([39]). Let $G = \langle g \rangle$ be a group with order $p^m > 4$ being p a prime and $m \in \mathbb{N}$ greater than or equal to 2. Let $\boldsymbol{A}$ be an aggregation operator and T a t-norm. The following statements are equivalent:*

(i) *$\boldsymbol{A}$ preserves T-subgroups of G on products.*
(ii) *$\boldsymbol{A} \gg T$.*

Theorem 4.7 *([39]). Let $\mathbb{Z}(p^\infty)$ be a Prüfer p-group for some prime number p, $\boldsymbol{A}$ an aggregation operator and T a t-norm. The following assertions are equivalent:*

(i) *$\boldsymbol{A}$ preserves T-subgroups of $\mathbb{Z}(p^\infty)$ on products.*
(ii) *$\boldsymbol{A} \gg T$.*

As a consequence of the preceding review of the known results, it remains to analyze the preservation of T-subgroups on sets when $G \in \mathcal{C}$ in order to complete the study of the aggregation of T-subgroups. This is the most difficult case and its study is the objective of the next section. The following result shows the case when we set T to be the minimum t-norm.

Theorem 4.8 *([3]). Let G be a non-trivial group. The following statements are equivalent:*

(i) *$G \in \mathcal{C}$.*
(ii) *Every aggregation function preserves the structure of T_M-subgroup on sets.*

In the next section, we will discuss the preservation of T-subgroups on sets when $T \neq T_M$. This requires the use of pseudo-p domination in some cases and type-k asymmetric domination in others. Note that every aggregation operator pseudo-p dominates T_M and type-k asymmetrically dominates T_M (see Remarks 3.5 and 3.6) so Theorem 4.8 is consistent with the previous statement.

5. Aggregation *on sets* of T-subgroups when the subgroup lattice of the ambient group is a chain

This section contains the main contribution of this manuscript. Our aim is to complete the study of the preservation of T-subgroups, obtaining a characterization of those aggregation operators that preserve T-subgroups on sets when the subgroup lattice of the ambient group G is a chain. It is clear that such a characterization must be a property that relates an aggregation operator $\boldsymbol{A}$ and the t-norm T. This relation is going to be weaker than domination since we know that $\boldsymbol{A} \gg T$ is a sufficient condition for $\boldsymbol{A}$ to preserve T-subgroups. In fact, the key properties in this case are pseudo-p domination and type-k asymmetric domination and they are indeed weaker than domination (see Fig. 4).

In this situation, it is necessary to distinguish different cases since the particular group over which we define the T-subgroups is crucial in order to know which property is involved in their preservation. Recalling Section 4, the only groups for which $Lat(G)$ is a chain are the cyclic groups with p^m elements and $\mathbb{Z}(p^\infty)$ for each prime number p. We will need the following distinction for the study of the preservation of T-subgroups on sets:

5.1 Groups with 2 and 3 elements.
5.2 Groups with prime order greater than 3.
5.3 Cyclic groups with order p^m, being p a prime and m an integer greater than or equal to 2.
5.4 Infinite Prüfer p-groups.

5.1. *Groups with 2 and 3 elements*

Given that cyclic groups of 2 and 3 elements are the most basic groups, it is unsurprising that any aggregation operator preserves T-subgroups of these groups on sets.

Theorem 5.1. *Let G be a group with order 2 or 3 and a t-norm T. Every aggregation operator $\boldsymbol{A}$ preserves T-subgroups of G on sets.*

Proof. From axiom G2 in Definition 2.8 and given n T-subgroups $\mu_1, \dots, \mu_n$, it is necessary for each $i \in \{1, \dots, n\}$ that:

$$\mu_i(z) = \begin{cases} 1 & z = e \\ \alpha_i & z \neq e \end{cases} \text{ with } \alpha_i \in [0,1].$$

It is easy to check that each μ_i is a T_M-subgroup. Moreover, Theorem 4.8 establishes that any aggregation operator $\boldsymbol{A}$ preserves T_M-subgroups of G on sets. Therefore, since $T_M \geq T$, given $a, b \in G$:

$$A\circ\boldsymbol{\mu}(ab) \geq T_M(A\circ\boldsymbol{\mu}(a), A\circ\boldsymbol{\mu}(b)) \geq T(A\circ\boldsymbol{\mu}(a), A\circ\boldsymbol{\mu}(b))$$

and we conclude that $\boldsymbol{A}$ preserves T-subgroups of G on sets for every aggregation operator $\boldsymbol{A}$. □

However, the situation is not so simple when we take more complex groups. In any other case there exist aggregation operators that does not preserve T-subgroups for a t-norm $T \neq T_M$. The next example supports this statement:

Example 5.2. Let $G \in C$ a group with more than 3 elements. In this case, we can always find $g \in G$ with $o(g) = p^m \geq 4$ being p a prime number and $m \in \mathbb{N}$. Given a t-norm $T \neq T_M$, there exist $\alpha \in]0,1[$ such that $1 > \alpha > T(\alpha,\alpha)$. We can define then the T-subgroup:

$$\mu(z) = \begin{cases} 1 & \text{if } z = e, \\ \alpha & \text{if } z \in \{g, g^{p^m-1}\}, \\ T(\alpha,\alpha) & \text{otherwise.} \end{cases}$$

It is easy to check that we have indeed a T-subgroup. Note that this is not a T_M-subgroup. Let us define also the aggregation operator $\boldsymbol{A}$ with the n-ary operators $A : [0,1]^n \to [0,1]$ such that:

$$A(x_1,\dots,x_n) = \begin{cases} 1 & \text{if } x_i \geq \alpha\ \forall i \in \{1,\dots,n\}, \\ 0 & \text{if } \exists i \in \{1,\dots,n\} \text{ such that } x_i < \alpha. \end{cases}$$

Now, if we take $a = b = g$ and $\mu_i = \mu$ for all $i \in \{1,\dots,n\}$, we have:

$$\begin{aligned} A\circ\boldsymbol{\mu}(ab) &= A\circ\boldsymbol{\mu}(gg) = A(T(\alpha,\alpha),\dots,T(\alpha,\alpha)) = 0 < 1 \\ &= T(A(\alpha,\dots,\alpha), A(\alpha,\dots,\alpha)) = T(A\circ\boldsymbol{\mu}(a), A\circ\boldsymbol{\mu}(b)). \end{aligned}$$

Hence, $\boldsymbol{A}$ does not preserve T-subgroups of G.

The previous example motivates us to search for a condition that characterizes the preservation of T-subgroups when G is a group with more than 3 elements.

5.2. Cyclic groups with prime order

The following theorem is one of the main results of the present manuscript. It states that an aggregation operator preserves T-subgroups of a group with prime order p on sets if and only if it pseudo-p dominates T. Therefore, the order of the group has a significant influence in deciding whether or not an aggregation operator preserves T-subgroups on sets in this situation.

Theorem 5.3. *Let G be a group with prime order $p \geq 5$, T a t-norm and $\boldsymbol{A}$ an aggregation operator. The following statements are equivalent:*

(i) *$\boldsymbol{A}$ preserves T-subgroups of G on sets.*
(ii) *$\boldsymbol{A} \gg_p T$.*

Proof. $(i) \Rightarrow (ii)$. Let us set $\overline{x} = (x_1,\dots,x_n), \overline{y} = (y_1,\dots,y_n) \in [0,1]^n$ such that $(\overline{x},\overline{y}) \in \tilde{S}_p$. That is, there exist $k \in \left\{1,\dots,\frac{p-3}{2}\right\}$ and $k' \in \{1,\dots,p-1\}$ with $kk' \equiv 1 \mod p$ such that for each $i \in \{1,\dots,n\}$:

$$\text{either } x_i \geq y_i \geq (x_i)_T^{(k)} \text{ or } y_i \geq x_i \geq (y_i)_T^{(\min\{k',p-k'-2\})}.$$

Let us build n T-subgroups $\mu_1,\dots,\mu_n$ and select $a, b \in G \setminus \{e\}$ such that $\mu_i(a) = x_i$, $\mu_i(b) = y_i$ and $\mu_i(ab) = T(x_i,y_i)$ for each $i \in \{1,\dots,n\}$. In this situation, since $\boldsymbol{A}$ preserves T-subgroups on sets:

$$\begin{aligned} A(T(x_1,y_1),\dots,T(x_n,y_n)) &= A\circ\boldsymbol{\mu}(ab) \geq T(A\circ\boldsymbol{\mu}(a), A\circ\boldsymbol{\mu}(b)) \\ &= T(A(x_1,\dots,x_n), A(y_1,\dots,y_n)) \end{aligned} \quad (9)$$

and this would end the proof obtaining that $\boldsymbol{A} \gg_p T$. Let us define each T-subgroup μ_i depending on the value of x_i and y_i.

For $k = 1$, the only possibility is that $k' = 1$ and $x_i = y_i$ for all $i \in \{1,\dots,n\}$. Consequently, taking $a = b$ and:

$$\mu_i(z) = \begin{cases} 1 & \text{if } z = e, \\ x_i & \text{if } z \in \{a, a^{p-1}\}, \\ T(x_i,x_i) & \text{otherwise,} \end{cases}$$

we get the desired T-subgroups.

Suppose now that $k \in \left\{ l \in \mathbb{N} \mid 2 \le l \le \frac{p-3}{2} \right\}$ (note that when $p = 5$ the previous set reduces to the empty set and therefore the following discussion is not applicable). Fix an arbitrary $a \in G \setminus \{e\}$ and take $b = a^k$. As a consequence, $a = b^{k'}$ where k' is the only value of the set $\{l \in \mathbb{N} \mid 2 \le l \le p-2\}$ such that $kk' \equiv 1 \mod p$. Note that $k' \neq p-1$ since $k \neq p-1$ and $(p-1)(p-1) \equiv 1 \mod p$.

We need to define different T-subgroups for each $i \in \{1,\dots,n\}$ depending on the relationship between x_i and y_i. We are going to write the elements of the group G as powers of a in some cases and as powers of b in others since the order of the group is prime so every element $g \in G \setminus \{e\}$ is a generator ($G = \langle a \rangle = \langle b \rangle$). It is necessary to distinguish the following cases as a consequence of the definition of the property $\succcurlyeq_p$:

(a) $x_i \ge y_i \ge (x_i)_T^{(k)}$.
(b) $y_i \ge x_i \ge (y_i)_T^{(\min\{k', p-k'-2\})}$ and $\min\{k', p-k'-2\} = k' < p-k'-2$. Hence we have that $k' \in \left\{ l \in \mathbb{N} \mid 2 \le l \le \frac{p-3}{2} \right\}$ (recall that the case where $k' = k = 1$ was already analyzed at the beginning of the proof).
(c) $y_i \ge x_i \ge (y_i)_T^{(\min\{k', p-k'-2\})}$ and $\min\{k', p-k'-2\} = p-k'-2 < k'$. Therefore, $k' \in \left\{ l \in \mathbb{N} \mid \frac{p-1}{2} \le l \le p-2 \right\}$. Moreover, it is easy to check that $\left(\frac{p-1}{2}\right)(p-2) \equiv 1 \mod p$. With this statement and given that $k \in \left\{ l \in \mathbb{N} \mid 2 \le l \le \frac{p-3}{2} \right\}$, we conclude that $k' \in \left\{ l \in \mathbb{N} \mid \frac{p+1}{2} \le l \le p-3 \right\}$. Therefore, $1 \le p-k'-2 \le \frac{p-5}{2}$. Now, we have to distinguish several cases again:
  (c1) $p-k'-2 = 1$.
  (c2) $2 \le p-k'-2 \le \frac{p-5}{2}$. Note that this case does not make sense when $p = 7$.

Note that if $k' = p-k'-2$, then $k' = \frac{p-2}{2} \notin \mathbb{N}$. Consequently, this distinction covers all the possible relations between x_i and y_i when $(\overline{x}, \overline{y}) \in \tilde{S}_p$. In the subsequent discussion, we propose one valid T-subgroup for each one of the enumerated cases.

(a) Whenever $x_i \ge y_i \ge (x_i)_T^{(k)}$, there exists $w_{i,1} \in \{l \in \mathbb{N} \mid 2 \le l \le k\}$ satisfying the chain of inequalities $(x_i)_T^{(w_{i,1}-1)} \ge y_i \ge (x_i)_T^{(w_{i,1})}$. This is due to the identity:

$$\left[(x_i)_T^{(k)}, x_i\right] = \bigcup_{i=2}^{k} \left[(x_i)_T^{(i-1)}, (x_i)_T^{(i)}\right].$$

Hence, we can define the T-subgroup:

$$\mu_i(z) = \begin{cases} 1 & \text{if } z = e, \\ (x_i)_T^{(u)} & \text{if } z \in \{a^u, a^{p-u}\} \text{ with } 1 \le u < w_{i,1}, \\ y_i & \text{if } z \in \{a^u, a^{p-u}\} \text{ with } w_{i,1} \le u \le k, \\ T(x_i, y_i) & \text{otherwise.} \end{cases}$$

This fuzzy set is a T-subgroup as a consequence of Lemma 2.12. Moreover, $\mu_i(a) = x_i$, $\mu_i(b) = \mu_i(a^k) = y_i$ and $\mu_i(ab) = \mu_i(a^{k+1}) = T(x_i, y_i)$.

(b) Let us suppose that $y_i \ge x_i \ge (y_i)_T^{(k')}$ and $k' \in \left\{ l \in \mathbb{N} \mid 2 \le l \le \frac{p-3}{2} \right\}$. Thus, there exists $w_{i,2} \in \left\{ l \in \mathbb{N} \mid 2 \le l \le k' \right\}$ such that $(y_i)_T^{(w_{i,2}-1)} \ge x_i \ge (y_i)_T^{(w_{i,2})}$. As before, we can ensure that the next function is a T-subgroup due to Lemma 2.12:

$$\mu_i(z) = \begin{cases} 1 & \text{if } z = e, \\ (y_i)_T^{(u)} & \text{if } z \in \{b^u, b^{p-u}\} \text{ with } 1 \le u < w_{i,2}, \\ x_i & \text{if } z \in \{b^u, b^{p-u}\} \text{ with } w_{i,2} \le u \le k', \\ T(x_i, y_i) & \text{otherwise.} \end{cases}$$

It holds again that $\mu_i(a) = \mu_i(b^{k'}) = x_i$, $\mu_i(b) = y_i$ and $\mu_i(ab) = \mu_i(b^{k'+1}) = T(x_i, y_i)$.

(c) Finally, let us consider $y_i \ge x_i \ge (y_i)_T^{(p-k'-2)}$ and $1 \le p-k'-2 \le \frac{p-5}{2}$.
  (c1) When $p-k'-2 = 1$, we have $x_i = y_i$ and using the next T-subgroup (see Lemma 2.12) we obtain the desired result:

$$\mu_i(z) = \begin{cases} 1 & \text{if } z = e, \\ x_i & \text{if } z \in \left\{b, b^{p-1}\right\}, \\ T(x_i, x_i) & \text{if } z \in \left\{b^2, b^{p-2}\right\}, \\ x_i & \text{if } z \in \left\{b^3, b^{p-3}\right\}, \\ T(x_i, x_i) & \text{otherwise.} \end{cases}$$

Additionally: $\mu_i(a) = \mu_i(b^{p-k'}) = \mu_i(b^3) = x_i = \mu_i(b)$ and $\mu_i(ab) = \mu_i(b^{k'+1}) = \mu_i(b^{p-k'-1}) = \mu_i(b^2) = T(x_i, x_i) = T(x_i, y_i)$.

(c2) If $2 \le p-k'-2 \le \frac{p-5}{2}$, we can find $w_{i,3} \in \left\{ l \in \mathbb{N} \mid 2 \le l \le p-k'-2 \right\}$ with $(y_i)_T^{(w_{i,3}-1)} \ge x_i \ge (y_i)_T^{(w_{i,3})}$. Recall that this case is not applicable when $p=7$. By Lemma 2.12, the fuzzy set:

$$\mu_i(z) = \begin{cases} 1 & \text{if } z = e, \\ (y_i)_T^{(u)} & \text{if } z \in \{b^u, b^{p-u}\} \text{ with } 1 \le u < w_{i,3}, \\ x_i & \text{if } z \in \{b^u, b^{p-u}\} \text{ with } w_{i,3} \le u \le p-k'-2, \\ T(x_i, y_i) & \text{if } z \in \{b^{p-k'-1}, b^{k'+1}\}, \\ x_i & \text{if } z \in \{b^{p-k'}, b^{k'}\}, \\ T(x_i, y_i) & \text{otherwise} \end{cases}$$

is T-subgroup and we will select it for this case. Clearly, $\mu_i(a) = \mu_i(b^{k'}) = x_i$, $\mu_i(b) = y_i$ and $\mu_i(ab) = \mu_i(b^{k'+1}) = T(x_i, y_i)$.

As a conclusion, for each $k \in \left\{1, \dots, \frac{p-3}{2}\right\}$ we have set $a, b \in G \setminus \{e\}$ and for each possible value of x_i and y_i, we have found a T-subgroup μ_i so that inequality (9) is satisfied.

$(ii) \Rightarrow (i)$. Given n T-subgroups $\mu_1, \dots, \mu_n$, we will show that:

$$A \circ \boldsymbol{\mu}(ab) \ge T(A \circ \boldsymbol{\mu}(a), A \circ \boldsymbol{\mu}(b)) \tag{10}$$

for all $a, b \in G$. With this purpose, for each $a, b \in G$, let us find $\overline{x} = (x_1, \dots, x_n), \overline{y} = (y_1, \dots, y_n) \in [0,1]^n$ such that the next conditions hold:

- $(\overline{x}, \overline{y}) \in \tilde{S}_p$. That is, there exists an integer $k \in \left\{1, \dots, \frac{p-3}{2}\right\}$ such that either $x_i \ge y_i \ge (x_i)_T^{(k)}$ or $y_i \ge x_i \ge (y_i)_T^{(\min\{k', p-k'-2\})}$ for all $i \in \{1, \dots, n\}$ where k' is the only number of the set $\{1, \dots, p-1\}$ satisfying $kk' \equiv 1 \mod p$.
- $x_i \ge \mu_i(a)$ and $y_i \ge \mu_i(b)$ for all $i \in \{1, \dots, n\}$ or vice versa, $y_i \ge \mu_i(a)$ and $x_i \ge \mu_i(b)$ for all $i \in \{1, \dots, n\}$.
- $\mu_i(ab) \ge T(x_i, y_i)$ for all $i \in \{1, \dots, n\}$.

With this selection of $\overline{x}$ and $\overline{y}$ the next expression would be true since $A \succcurlyeq_p T$ by hypotheses:

$$\begin{aligned} A \circ \boldsymbol{\mu}(ab) &\ge A(T(x_1, y_1), \dots, T(x_n, y_n)) \\ &\ge T(A(x_1, \dots, x_n), A(y_1, \dots, y_n)) \ge T(A \circ \boldsymbol{\mu}(a), A \circ \boldsymbol{\mu}(b)) \end{aligned} \tag{11}$$

and the proof would be complete.

Given some arbitrary $a, b \in G$, let us find $(x_1, \dots, x_n)$, $(y_1, \dots, y_n)$ and k satisfying the above mentioned requirements. Let us first note that if a, b or ab are the neutral element of G, the inequality (10) is trivially satisfied. Let us suppose then that $a, b, ab \in G \setminus \{e\}$. Since G is a cyclic group of prime order $G = \langle a \rangle = \langle b \rangle$ and there must exist $s, s' \in \{1, \dots, p-1\}$ such that $b = a^s$ and $a = b^{s'}$ with $ss' \equiv 1 \mod p$. In fact, $a = a^{ss'}$. It is necessary to consider different cases depending on the values of s and s':

(a) If s or s' belong to the set $\left\{\frac{p-1}{2}, p-2\right\}$, then it is easy to check that $\mu_i(ab) \in \{\mu_i(a), \mu_i(b)\}$ for all $i \in \{1, \dots, n\}$. Therefore $A \circ \boldsymbol{\mu}(ab) \in \{A \circ \boldsymbol{\mu}(a), A \circ \boldsymbol{\mu}(b)\}$ and Inequality (10) always hold. Note that it is not necessary to select any $(x_1, \dots, x_n), (y_1, \dots, y_n)$ in this case.

(b) If any of the values s or s' are equal to $p-1$ it is clear that $\mu_i(ab) = \mu_i(e) = 1$ for all $i \in \{1, \dots, n\}$ and hence $A \circ \boldsymbol{\mu}(ab) = 1 \ge T(A \circ \boldsymbol{\mu}(a), A \circ \boldsymbol{\mu}(b))$.

(c) If $s, s' \in \left\{ l \in \mathbb{N} \mid 1 \le l \le \frac{p-3}{2} \right\}$, then $\min\{s', p-s'-2\} = s'$. In this situation we can take $k = s$, $k' = s'$, $x_i = \mu_i(a)$ and $y_i = \mu_i(b)$ for all $i \in \{1, \dots, n\}$. Whenever $\mu_i(a) \ge \mu_i(b)$, we have:

$$x_i = \mu_i(a) \ge y_i = \mu_i(b) = \mu_i(a^s) \ge \mu_i(a)_T^{(s)} = (x_i)_T^{(k)}.$$

In addition, if $\mu_i(b) \ge \mu_i(a)$, we get that:

$$y_i = \mu_i(b) \ge x_i = \mu_i(a) = \mu_i(b^{s'}) \ge \mu_i(b)_T^{(s')} = (y_i)_T^{(k')}.$$

Moreover, $\mu_i(ab) \ge T(\mu_i(a), \mu_i(b)) = T(x_i, y_i)$ since μ_i is a T-subgroup. Thus, this selection meets all the conditions required for Inequality (11) to be true.

Notice that cases (d) and (e) do not make sense when $p=5$ so we do not consider them in that situation.

(d) If $s \in \left\{ l \in \mathbb{N} \mid 1 \le l \le \frac{p-3}{2} \right\}$ and $s' \in \left\{ l \in \mathbb{N} \mid \frac{p+1}{2} \le l \le p-3 \right\}$ we need to note first that, since $ss' \equiv 1 \mod p$, then $s=1$ if and only if $s'=1$. Therefore, in this case $2 \le s \le \frac{p-3}{2}$. Note that $\min\{s', p-s'-2\} = p-s'-2 \in \left\{ l \in \mathbb{N} \mid 1 \le l \le \frac{p-5}{2} \right\}$ and that:

$$\mu_i(ab) = \mu_i(b^{s'+1}) = \mu_i((b^{-1})^{p-s'-1}) \geq \mu_i(b^{-1})_T^{(p-s'-1)} = \mu_i(b)_T^{(p-s'-1)}. \tag{12}$$

Let us set $k = s$ and $k' = s'$ and let us select different values of x_i and y_i for each $i \in \{1, \dots, n\}$ depending on μ_i:

1. If $\mu_i(ab) > \max\{\mu_i(a), \mu_i(b)\}$, we just take $x_i = y_i = \mu_i(ab)$.
2. Whenever $\mu_i(a) > \mu_i(b)$ and $\mu_i(a) \geq \mu_i(ab)$, we can select $x_i = \mu_i(a)$ and $y_i = \mu_i(b)$ to have:
$$x_i = \mu_i(a) > y_i = \mu_i(b) = \mu_i(a^s) \geq \mu_i(a)_T^{(s)} = (x_i)_T^{(k)}.$$
3. If $\mu_i(b) \geq \max\{\mu_i(ab), \mu_i(a)\}$, as a consequence of (12) there exists $w_i \in \{2, \dots, p - s' - 1\}$ with:
$$\mu_i(b)_T^{(w_i-1)} \geq \mu_i(ab) \geq \mu_i(b)_T^{(w_i)} \geq \mu_i(b)_T^{(p-s'-1)}.$$
Hence:
   - Whenever $\mu_i(a) \geq \mu_i(b)_T^{(w_i-1)}$, we can take $x_i = \mu_i(a)$ and $y_i = \mu_i(b)$. Thus:
$$y_i = \mu_i(b) \geq x_i = \mu_i(a) \geq \mu_i(b)_T^{(w_i-1)} \geq \mu_i(b)_T^{(p-s'-2)} = (y_i)_T^{(p-s'-2)}.$$
   - Otherwise, if $\mu_i(b)_T^{(w_i-1)} > \mu_i(a)$ we will take $x_i = \mu_i(b)_T^{(w_i-1)}$ and $y_i = \mu_i(b)$ so that:
$$y_i = \mu_i(b) \geq x_i = \mu_i(b)_T^{(w_i-1)} \geq \mu_i(b)_T^{(p-s'-2)} = (y_i)_T^{(p-s'-2)},$$
   and $\mu_i(ab) \geq \mu_i(b)_T^{(w_i)} = T(\mu_i(b)_T^{(w_i-1)}, \mu_i(b)) = T(x_i, y_i)$

Note that, in any case $x_i \geq \mu_i(a)$, $y_i \geq \mu_i(b)$ and $\mu_i(ab) \geq T(x_i, y_i)$.

(e) If $s \in \left\{ l \in \mathbb{N} \mid \frac{p+1}{2} \leq l \leq p-3 \right\}$ and $s' \in \left\{ l \in \mathbb{N} \mid 1 \leq l \leq \frac{p-3}{2} \right\}$, we have to interchange the values of k and k' with respect to the previous cases. That is $k = s'$ and $k' = s$. With this change we can carry out a similar procedure as in case (c) interchanging a with b and s with s' in all the proof, and hence it is an analogous case.

(f) Finally, let us suppose that $s, s' \in \left\{ l \in \mathbb{N} \mid \frac{p+1}{2} \leq l \leq p-3 \right\}$. Since $s = \frac{p+1}{2}$ if and only if $s' = 2$ and vice versa, we have to consider the restriction $s, s' \in \left\{ l \in \mathbb{N} \mid \frac{p+3}{2} \leq l \leq p-3 \right\}$ (note that if $p \in \{5, 7\}$ this situation does not apply). We will use the values $k = p - s$, $k' = p - s'$, $x_i = \mu_i(a)$ and $y_i = \mu_i(b)$ for all $i \in \{1, \dots, n\}$. With this selection, we can check that $k, k' \in \left\{ l \in \mathbb{N} \mid 3 \leq l \leq \frac{p-3}{2} \right\}$ with $kk' \equiv p^2 - ps - ps' + ss' \equiv ss' \equiv 1 \mod p$. Hence, $\min\{k', p - k' - 2\} = k'$. In addition $b = a^s = (a^{-1})^{p-s}$ and $a^{-1} = b^{p-s'}$. Whenever $\mu_i(a) \geq \mu_i(b)$, it is clear that:
$$x_i = \mu_i(a) \geq y_i = \mu_i(b) = \mu_i(a^{p-s}) \geq \mu_i(a)_T^{(p-s)} = (x_i)_T^{(p-s)} = (x_i)_T^{(k)}.$$
In the converse situation where $\mu_i(b) > \mu_i(a)$ we have:
$$y_i = \mu_i(b) > x_i = \mu_i(a) = \mu_i(b^{p-s'}) \geq \mu_i(b)_T^{(p-s')} = (y_i)_T^{(p-s')}$$
$$= (y_i)_T^{(k')} = (y_i)_T^{(\min\{k', p-k'-2\})}.$$
Obviously $\mu_i(ab) \geq T(\mu_i(a), \mu_i(b)) = T(x_i, y_i)$.

Note that in all cases the selected $(x_1, \dots, x_n), (y_1, \dots, y_n)$ satisfy the conditions given at the beginning of the proof so that Inequality (11) holds. Therefore, this ends the proof. Note that the cases (a)-(f) cover all the possible values for s, s'. □

We can now compare the known results about the preservation of T-subgroups on both sets and products for this kind of groups. Theorem 4.5 establishes that type-k domination characterizes the preservation of T-subgroups on products for groups with a prime number of elements. Furthermore, the previous result states that the aggregation operators that preserve T-subgroups of these groups (except for groups with 2 or 3 elements) pseudo-p dominate T. According to Remark 3.8, we know that these two properties are not equivalent in general. Therefore, we can present the following remark.

Remark 5.4. Let G be a group of prime order and T a t-norm. If an aggregation operator preserves T-subgroups of G on sets, it does not necessarily preserve T-subgroups of G on products.

5.3. Groups with order p^m where p is a prime and $m \in \mathbb{N}$ is greater than 1

Let us now analyze the rest of the finite groups where $Lat(G)$ is a chain. The cyclic group of four elements must be studied separately because it differs significantly from the other cases in this subsection.

Theorem 5.5. *Let $G = \langle g \rangle$ be the cyclic group with 4 elements, $\boldsymbol{A}$ an aggregation operator and T a t-norm. The following statements are equivalent:*

(i) *$\boldsymbol{A}$ preserves the structure of T-subgroup of G on sets.*

(ii) $\boldsymbol{A} \succcurlyeq_5 T$.

Proof. $(i) \Rightarrow (ii)$ Let $\overline{x} = (x_1, \dots, x_n) \in [0,1]^n$. By Theorem 2.10, for each $i \in \{1, \dots, n\}$ the fuzzy set:

$$\mu_i(z) = \begin{cases} 1 & \text{if } z = e, \\ x_i & \text{if } z \in \{g, g^{-1}\}, \\ T(x_i, x_i) & \text{otherwise} \end{cases}$$

is T-subgroup. Consequently, if we take $a = b = g$ we have that $ab = g^2$, $\mu_i(a) = \mu_i(b) = x_i$ and $\mu_i(ab) = T(x_i, x_i)$. Thus, given that $\boldsymbol{A}$ preserves T-subgroups:

$$\begin{aligned} A(T(x_1, x_1), \dots, T(x_n, x_n)) = A \circ \boldsymbol{\mu}(ab) &\geq T(A \circ \boldsymbol{\mu}(a), A \circ \boldsymbol{\mu}(b)) \\ &= T(A(x_1, \dots, x_n), A(x_1, \dots, x_n)), \end{aligned}$$

which proves that $\boldsymbol{A} \succcurlyeq_5 T$ by Definition 3.3 since:

$$\tilde{S}_5 = \left\{(\overline{x}, \overline{y}) \in [0,1]^n \times [0,1]^n \mid \overline{x} = \overline{y}\right\} = \left\{(\overline{x}, \overline{x}) \mid \overline{x} \in [0,1]^n\right\}.$$

$(ii) \Rightarrow (i)$. Let us suppose that $\boldsymbol{A} \succcurlyeq_5 T$ and take the T-subgroups $\mu_1, \dots, \mu_n$. The goal here is to check that $A \circ \boldsymbol{\mu}$ fulfills the axiom G3 of T-subgroup. With this purpose, let us take $a, b \in G$. If any of the elements a, b or ab are equal to the neutral element $e \in G$, the inequality $A \circ \boldsymbol{\mu}(ab) \geq T(A \circ \boldsymbol{\mu}(a), A \circ \boldsymbol{\mu}(b))$ trivially holds. Therefore, let us suppose $a, b, ab \notin \{e\}$. Consequently, $a \neq b^{-1}$. Moreover, let us consider two cases:

(a) If $a \neq b$ then we have that $\{a, a^{-1}\} \cap \{b, b^{-1}\} = \emptyset$ and we can suppose without loss of generality that $G = \{e, a, a^{-1}, b\}$. Hence, $ab \in \{a, a^{-1}\}$ or $ab = b$. Additionally, either $\mu_i(ab) = \mu_i(a)$ for all $i \in \{1, \dots, n\}$ or $\mu_i(ab) = \mu_i(b)$ for all $i \in \{1, \dots, n\}$. Therefore, $A \circ \boldsymbol{\mu}(ab) \in \{A \circ \boldsymbol{\mu}(a), A \circ \boldsymbol{\mu}(b)\}$ and G3 holds.
(b) If $a = b$, clearly $\mu_i(a) = \mu_i(b)$ for all $i \in \{1, \dots, n\}$. Taking $(x_1, \dots, x_n) \in [0,1]^n$ such that $x_i = \mu_i(a) = \mu_i(b)$, the inequality $\mu_i(ab) \geq T(x_i, x_i)$ holds. Moreover, since $\boldsymbol{A} \succcurlyeq_5 T$:

$$\begin{aligned} A \circ \boldsymbol{\mu}(ab) &\geq A(T(x_1, x_1), \dots, T(x_n, x_n)) \geq T(A(x_1, \dots, x_n), A(x_1, \dots, x_n)) \\ &= T(A \circ \boldsymbol{\mu}(a), A \circ \boldsymbol{\mu}(b)) \end{aligned}$$

In both cases we can ensure the preservation of T-subgroups on sets. □

Using the same reasoning as for Remark 5.4, we can present the following result.

Remark 5.6. Let G be a cyclic group with 4 elements and T a t-norm. If an aggregation operator preserves T-subgroups of G on sets, it does not necessarily preserve T-subgroups of G on products.

As another direct consequence of Theorem 5.5 note that the aggregation operators that preserve T-subgroups on the cyclic group of 4 elements are the same as the ones preserving T-subgroups of the cyclic group of 5 elements (see Theorem 5.3). This is the only case where the preservation of T-subgroups in a group of prime order is equivalent to that of a group with order p^m where $m > 1$. Indeed, in the following results we will show that type-k asymmetric domination is the property that holds for the remaining groups in $\mathcal{C}$. Let us start with finite groups.

Theorem 5.7. *Let $G = \langle g \rangle$ be a group with $|G| = r = p^m > 4$ for an integer $m \geq 2$ and a prime p. Let $\boldsymbol{A}$ be an aggregation operator and T a t-norm. The following propositions are equivalent:*

(i) *$\boldsymbol{A}$ preserves the structure of T-subgroup of G on sets.*
(ii) $\boldsymbol{A} \gg^{\frac{p^m - p}{2}} T$.

Proof. $(i) \Rightarrow (ii)$. Let $\overline{x} = (x_1, \dots, x_n), \overline{y} = (y_1, \dots, y_n) \in [0,1]^n$ be two n-tuples such that $(\overline{x}, \overline{y}) \in S^{\frac{p^m - p}{2}}$. That is $y_i \geq (x_i)_T^{\left(\frac{p^m - p}{2}\right)}$ for all $i \in \{1, \dots, n\}$. Let us define n T-subgroups $\mu_1, \dots, \mu_n$. Each μ_i will depend on the relation between x_i and y_i. On the one hand, if $y_i \geq x_i$ we will take:

$$\mu_i(z) = \begin{cases} 1 & \text{if } z = e, \\ y_i & \text{if } z \in \langle g^p \rangle \setminus \{e\}, \\ x_i & \text{if } z \in \{g, g^{-1}\}, \\ T(x_i, y_i) & \text{otherwise,} \end{cases}$$

which is a T-subgroup as a consequence of a similar procedure as the one given in Theorem 4.20 of [39]. If we consider $a = g$ and $b = g^{\frac{p^m-p}{2}} = g^{p\left(\frac{p^{m-1}-1}{2}\right)} \in \langle g^p\rangle \setminus \{e\}$, then $ab \notin \langle g^p\rangle \cup \{g, g^{-1}\}$. Thus $\mu_i(a) = x_i,\ \mu_i(b) = y_i$ and $\mu_i(ab) = T(x_i, y_i)$.

On the other hand, whenever $x_i > y_i \geq (x_i)_T^{\left(\frac{p^m-p}{2}\right)}$ there exists $v_i \in \left\{2, \dots, \frac{p^m-p}{2}\right\}$ such that $(x_i)_T^{(v_i-1)} \geq y_i \geq (x_i)_T^{(v_i)}$. Hence, we can define the next fuzzy set:

$$\mu_i(z) = \begin{cases} 1 & \text{if } z = e, \\ (x_i)_T^{(u)} & \text{if } z \in \{g^u, g^{r-u}\} \text{ with } 1 \leq u < v_i, \\ y_i & \text{if } z \in \{g^u, g^{r-u}\} \text{ with } v_i \leq u \leq \frac{p^m-p}{2}, \\ T(x_i, y_i) & \text{otherwise.} \end{cases}$$

By Lemma 2.12 this is a T-subgroup. Again, in this case $\mu_i(a) = x_i,\ \mu_i(b) = y_i$ and $\mu_i(ab) = T(x_i, y_i)$.

Finally, since $\boldsymbol{A}$ preserves T-subgroups of G on sets:

$$\begin{aligned} A(T(x_1, y_1), \dots, T(x_n, y_n)) &= A \circ \boldsymbol{\mu}(ab) \geq T(A \circ \boldsymbol{\mu}(a), A \circ \boldsymbol{\mu}(b)) \\ &= T(A(x_1, \dots, x_n), A(y_1, \dots, y_n)). \end{aligned}$$

Consequently, $\boldsymbol{A} \gg^{\frac{p^m-p}{2}} T$.

$(ii) \Rightarrow (i)$. Let a and b be arbitrary elements of G and let us prove that $A \circ \boldsymbol{\mu}(ab) \geq T(A \circ \boldsymbol{\mu}(a), A \circ \boldsymbol{\mu}(b))$. If $a = e$ or $b = e$, this inequality trivially holds so let us suppose that $a, b \in G \setminus \{e\}$. Since $G \in C$ and without loss of generality, we can state that $\langle b\rangle \subseteq \langle a\rangle \subseteq G$. We have to consider two different cases depending on the order of a and b.

If $o(a) \neq p^m$, we have that $o(a) = p^{m'}$ with $1 \leq m' < m$. Hence, $b = a^s$ where $s \in \{1, \dots, p^{m'} - 1\}$. It is easy to check that:

$$\frac{p^m - p}{2} \geq \frac{p^{m'} - 1}{2} \quad \text{and} \quad \frac{p^m - p}{2} \geq \frac{p^{m'}}{2}.$$

Consequently, we have $\mu_i(b) \geq \mu_i(a)_T^{\left(\frac{p^{m'}-1}{2}\right)} \geq \mu_i(a)_T^{\left(\frac{p^m-p}{2}\right)}$ if $p \neq 2$ and $\mu_i(b) \geq \mu_i(a)_T^{\left(\frac{p^{m'}}{2}\right)} \geq \mu_i(a)_T^{\left(\frac{p^m-p}{2}\right)}$ if $p = 2$. This deduction is due to the fact that $\mu_i(b) = \mu_i(a^s) = \mu_i\left(a^{p^{m'}-s}\right)$, and $\min\{s, p^{m'} - s\} \leq \frac{p^{m'}-1}{2}$ when $p \neq 2$ and $\min\{s, p^{m'} - s\} \leq \frac{p^{m'}}{2}$ when $p = 2$. In this case, we will take $x_i = \mu_i(a)$ and $y_i = \mu_i(b)$ for all $i \in \{1, \dots, n\}$.

Let us now suppose that $o(a) = p^m$. Thus, $b = a^s$ for some $s \in \{1, \dots, p^m - 1\}$. Note that, whenever $s \in \left\{ l \in \mathbb{N} \mid 1 \leq l \leq \frac{p^m-p}{2} \text{ or } \frac{p^m+p}{2} \leq l \leq p^m - 1\right\}$ the next inequality holds:

$$\mu_i(b) \geq \mu_i(a)_T^{\left(\frac{p^m-p}{2}\right)}$$

and we will select again $x_i = \mu_i(a)$ and $y_i = \mu_i(b)$ for all $i \in \{1, \dots, n\}$. The case where $s \in \left\{ l \in \mathbb{N} \mid \frac{p^m-p}{2} + 1 \leq l \leq \frac{p^m+p}{2} - 1 \right\}$ is a bit more involved.

In this case, we rewrite $s = \frac{p^m-p}{2} + j$ with $j \in \{1, \dots, p-1\}$. Therefore, the greatest common divisor between $o(a)$ and s is $d = \gcd(o(a), s) = \gcd\left(p^m, \frac{p^m-p}{2} + j\right) = 1$ and we can conclude that $o(b) = o(a^s) = p^m$. If d was different from 1 we would have $d = p^{m_1}$ with $1 \leq m_1 \leq m$. But in that case $p^{m_1} \mid \left(\frac{p^m-p}{2} + j\right)$. Hence, $p|j$ which is a contradiction since $p > j$.

As a consequence of the previous discussion, the only remaining possibility is that $\langle a\rangle = \langle b\rangle = G$. We can suppose then that there exists $s' \in \{1, \dots, p^m - 1\}$ such that $a = b^{s'}$. In particular, $b = a^s = (b^{s'})^s = b^{ss'}$, obtaining that $ss' \equiv 1 \mod p^m$. Let us check that, since $s \in \left\{ l \in \mathbb{N} \mid \frac{p^m-p}{2} + 1 \leq l \leq \frac{p^m+p}{2} - 1 \right\}$, it is impossible for s' to be in that same set. In order to get a contradiction. Let us suppose that $s' \in \left\{ l \in \mathbb{N} \mid \frac{p^m-p}{2} + 1 \leq l \leq \frac{p^m+p}{2} - 1 \right\}$. We need to study separately when $p = 2$ and $p \neq 2$.

If $p = 2$, the only possibility is $s = 2^{m-1} = s'$. However, 2^m would divide $ss' - 1 = 2^{2m-2} - 1$ deriving in a contradiction because an even number would divide an odd one.

For any $p \neq 2$, we can write $s = \frac{p^m-1}{2} + j$ and $s' = \frac{p^m-1}{2} + j'$ with $j, j' \in \{-\frac{p-3}{2}, \dots, \frac{p-1}{2}\}$. Moreover, $p^m | ss' - 1$ i.e. there exist $l \in \mathbb{Z}$ such that:

$$\begin{aligned} &lp^m = \left(\left(\frac{p^m-1}{2} + j\right)\left(\frac{p^m-1}{2} + j'\right)\right) - 1 \Leftrightarrow \\ &lp^m = \frac{p^{2m} - 2p^m + 1}{4} + (j + j')\left(\frac{p^m - 1}{2}\right) + jj' - 1 \Leftrightarrow \\ &4lp^m = p^{2m} - 2p^m + 1 + 2jp^m + 2j'p^m - 2j - 2j' + 4jj' - 4 \Leftrightarrow \\ &p^m(4l - p^m + 2 - 2j - 2j') = p^m\bar{l} = 4jj' - 2j - 2j' - 3, \end{aligned}$$

where $\bar{l} = 4l - p^m + 2 - 2j - 2j' \in \mathbb{Z}$. Therefore, $p^m | 4jj' - 2j - 2j' - 3$. Note that $-(p-3)(p-1) \le 4jj' \le (p-1)^2$ and $-(p-1) \le -2j, -2j' \le p-3$ and hence since $p^m \bar{l} = 4jj' - 2j - 2j' - 3$ we have that:

$$-p^m \le -p^2 < -p^2 + 2p - 4 \le p^m \bar{l} \le p^2 - 8 < p^2 \le p^m.$$

The only possibility then is that $\bar{l} = 0$ and it leads to the equation $4jj' - 2j - 2j' = 3$. Here we get our desired contradiction because we are stating that $2|3$.

As a consequence of the previous discussion, $a = b^{s'}$ where s' is not contained in $\left\{ l \in \mathbb{N} \mid \frac{p^m - p}{2} + 1 \le l \le \frac{p^m + p}{2} - 1 \right\}$. Therefore, $\mu_i(a) \ge \mu_i(b)_T^{(\frac{p^m-p}{2})}$. Now, we will fix $x_i = \mu_i(b)$ and $y_i = \mu_i(a)$ for all $i \in \{1, \dots, n\}$.

With the previous selection for x_i and y_i in each one of the cases it is clear that $\mu_i(ab) \ge T(x_i, y_i)$ and thus:

$$\begin{aligned} A \circ \boldsymbol{\mu}(ab) &\ge A(T(x_1, y_1), \dots, T(x_n, y_n)) \\ &\ge T(A(x_1, \dots, x_n), A(y_1, \dots, y_n)) = T(A \circ \boldsymbol{\mu}(a), A \circ \boldsymbol{\mu}(b)). \end{aligned}$$

Since a and b are arbitrary elements of G and $\mu_1, \dots, \mu_n$ are arbitrary T-subgroups of G, the last inequality proves that $\boldsymbol{A}$ preserves T-subgroups of G on sets. □

This result shows that the conditions for the preservation of T-subgroups of non-simple finite cyclic groups of order greater than four on sets and products are not equivalent. Recall that in the case being studied, all aggregation operators that preserve T-subgroups on products dominate the t-norm (see Theorem 4.6). As domination and type-k asymmetric domination are not equivalent properties (see Remark 3.8) the following remark is given.

Remark 5.8. Let G be a cyclic group with $p^m > 4$ elements where p is a prime number and $m \ge 2$. Let T be a t-norm. An aggregation operator preserving T-subgroups of G on sets does not necessarily preserve T-subgroups of G on products.

5.4. Prüfer p-groups

Infinite groups with a subgroup lattice that forms a chain behave similarly to the groups discussed in the previous section regarding the preservation of T-subgroups on sets. Type-∞ asymmetric domination characterizes the preservation in this case. To show this assertion, we need to establish an auxiliary lemma:

Lemma 5.9. *Let T be a t-norm and G a group such that there exists $g \in G$ with $o(g) = r > 5$. Given $x, y \in [0,1]$ and $v \in \left\{2, \dots, \left[\frac{r}{2}\right] - 1\right\}$ with $x \ge x_T^{(v-1)} \ge y \ge x_T^{(v)} \ge x_T^{([\frac{r}{2}]-1)}$. Then, the fuzzy set:*

$$\mu(z) = \begin{cases} 1 & \text{if } z = e, \\ (x)_T^{(u)} & \text{if } z \in \{g^u, g^{r-u}\} \text{ with } 1 \le u < v, \\ y & \text{if } z \in \{g^u, g^{r-u}\} \text{ with } u = v, \\ \alpha_u & \text{if } z \in \{g^u, g^{r-u}\} \text{ with } v+1 \le u \le \left[\frac{r}{2}\right], \\ T(x,y) & \text{otherwise,} \end{cases} \tag{13}$$

where in the last case α_u can be any number in the interval $[T(x,y), y]$, is a T-subgroup.

Proof. This result is an extension of Lemma 2.12. The fuzzy set (13) trivially satisfies the axioms G1 and G2 of T-subgroup. It only remains to check that $\mu(z_1 z_2) \ge T(\mu(z_1), \mu(z_2))$ for all $z_1, z_2 \in G$. Since $\langle g \rangle$ is a cyclic group of finite order r, if $z_1, z_2 \in \langle g \rangle$ the inequality G3 holds due to Lemma 2.12. Moreover, if $z_1 \in G \setminus \langle g \rangle$, then $\mu(z_1) = T(x,y)$ and it is straightforward that $\mu(z_1 z_2) \ge T(x,y) \ge T(\mu(z_1), \mu(z_2))$ for any $z_2 \in G$. We have the same situation when $z_2 \in G \setminus \langle g \rangle$. Therefore, μ satisfies G3 and it is a T-subgroup. □

The following theorem completes the study of the preservation of T-subgroups on sets.

Theorem 5.10. *Let T be a t-norm, $\boldsymbol{A}$ an aggregation operator and p a prime number. The following statements are equivalent:*

(i) *$\boldsymbol{A}$ preserves T-subgroups of $\mathbb{Z}(p^\infty)$ on sets.*
(ii) $\boldsymbol{A} \gg^\infty T$.

Proof. $(i) \Rightarrow (ii)$. Let us suppose that $\boldsymbol{A}$ preserves T-subgroups and let us consider $\overline{x} = (x_1, \dots, x_n), \overline{y} = (y_1, \dots, y_n) \in [0,1]$ such that $(\overline{x}, \overline{y}) \in S^\infty$. That is, there exist $k \in \mathbb{N}$ with $y_i \ge (x_i)_T^{(k)}$. We can find $m \in \mathbb{N}$ such that $k \le \frac{p^m - p}{2}$. Now, since $\mathbb{Z}(p^\infty)$ contains a copy of

$\mathbb{Z}_{p^m}$ we have an element $g \in \mathbb{Z}(p^\infty)$ with order p^m. Hence, we will define a T-subgroup μ_i according to the value of x_i and y_i for each $i \in \{1, \ldots, n\}$. If $y_i \geq x_i$ the corresponding T-subgroup will be:

$$\mu_i(z) = \begin{cases} 1 & \text{if } z = e, \\ y_i & \text{if } z \in \langle g^p \rangle \setminus \{e\}, \\ x_i & \text{if } z \in \{g, g^{-1}\}, \\ T(x_i, y_i) & \text{otherwise.} \end{cases}$$

We know that this is a T-subgroup due to the proof in Theorem 5.7. The only other possibility is $x_i > y_i \geq (x_i)_T^{(k)}$ in which case there exist $w_i \in \{2, \ldots k\}$ such that $(x_i)_T^{(w_i-1)} \geq y_i \geq (x_i)_T^{(w_i)}$. We can take:

$$\mu_i(z) = \begin{cases} 1 & \text{if } z = e, \\ (x_i)_T^{(u)} & \text{if } z \in \{g^u, g^{r-u}\} \text{ with } 1 \leq u < w_i, \\ y_i & \text{if } z \in \{g^u, g^{r-u}\} \text{ with } w_i \leq u \leq \frac{p^m - p}{2}, \\ T(x_i, y_i) & \text{otherwise.} \end{cases}$$

By Lemma 5.9, μ_i is a T-subgroup.

The next step is to take $a = g$ and $b = g^{\frac{p^m-p}{2}}$ so that $\mu_i(a) = x_i$, $\mu_i(b) = y_i$ and $\mu_i(ab) = T(x_i, y_i)$ for all $i \in \{1, \ldots, n\}$. Thus:

$$\begin{aligned} A(T(x_1, y_1), \ldots, T(x_n, y_n)) &= A \circ \boldsymbol{\mu}(ab) \geq T(A \circ \boldsymbol{\mu}(a), A \circ \boldsymbol{\mu}(b)) \\ &= T(A(x_1, \ldots, x_n), A(y_1, \ldots, y_n)) \end{aligned}$$

and $\boldsymbol{A} \gg^\infty T$.

$(ii) \Rightarrow (i)$. First recall that $\mathbb{Z}(p^\infty) \in \mathcal{C}$ so given $a, b \in \mathbb{Z}(p^\infty)$ we can suppose without loss of generality that $\langle b \rangle \subseteq \langle a \rangle$. All the elements in a Prüfer p-group have finite order so there exists $s \in \{1, \ldots, o(a) - 1\}$ such that $b = a^s$. Given n T-subgroups of $\mathbb{Z}(p^\infty)$ denoted by $\mu_1, \ldots, \mu_n$, we will have for all $i \in \{1, \ldots, n\}$ that $\mu_i(b) \geq \mu_i(a)_T^{(s)} \geq \mu_i(a)_T^{(o(a))}$. Therefore, if we take $k = o(a)$, $x_i = \mu_i(a)$ and $y_i = \mu_i(b)$ we will have $\mu_i(ab) = T(x_i, y_i)$ and $y_i \geq (x_i)_T^{(k)}$. As the hypothesis is $\boldsymbol{A} \gg^\infty T$:

$$\begin{aligned} A \circ \boldsymbol{\mu}(ab) &= A(T(x_1, y_1), \ldots, T(x_n, y_n)) \\ &\geq T(A(x_1, \ldots, x_n), A(y_1, \ldots, y_n)) = T(A \circ \boldsymbol{\mu}(a), A \circ \boldsymbol{\mu}(b)) \quad \square \end{aligned}$$

With the same reasoning as for Remark 5.8, we can obtain the next result.

Remark 5.11. Let p be a prime number. An aggregation operator preserving T-subgroups of $\mathbb{Z}(p^\infty)$ on sets does not necessarily preserve T-subgroups of $\mathbb{Z}(p^\infty)$ on products.

6. Comparison of the results obtained for aggregation *on products* and *on sets*

In this section we will compare the results obtained for the preservation of T-subgroups on products with those given for the preservation of T-subgroups on sets. We will detail in which cases the aggregation operators preserving T-subgroups on products are the same as those preserving T-subgroups on sets. Moreover, we will examine cases where the preservation on sets (on products) of T-subgroups of one specific group implies the preservation on sets (on products) of T-subgroups of other different groups.

By combining the remarks of Section 5 with Proposition 4.2, we can derive that if an aggregation operator preserves T-subgroups of a group G whose subgroup lattice is a chain, then it also preserves T-subgroups of G on sets and that the converse implication is not true in general.

Fig. 5 shows the known relations between the preservation of T-subgroups on both products and sets for all possible groups. For simplicity, $\mathbb{Z}_r$ denotes the cyclic group of order $r \in \mathbb{N}$. The interpretation of this diagram is simple: for a fixed t-norm T, if an aggregation operator preserves the structure of T-subgroup of the group at the origin of an arrow ($\Rightarrow$), then it also preserves T-subgroups of the group at the end of that arrow. We use a different type of arrow ($\rightarrow$) entering inside the boxes to denote which exact property characterizes the preservation of T-subgroups of each group. The left box of the diagram contains the relations between the preservation of T-subgroups on products for the different groups. The right panel includes the known relations between the preservation of T-subgroups on sets for the same groups. This is the reason why each of the groups is shown twice in the diagram. Therefore, the main results obtained in this work and in [3,39] are summarized in Fig. 5.

In addition, by examining Fig. 5, it can be deduced from Remark 3.15 that an infinite number of aggregation operators preserving T-subgroups on sets for those groups whose subgroup lattice is a chain can be considered when the t-norm T is part of Dubois Prade or Mayor-Torrens family. However these same aggregation operators do not preserve T-subgroups on products for the same t-norms and ambient groups. Note that the aforementioned remark concerns the behavior of the t-norms in these families under different types of domination which precisely characterize the aggregation on sets and on products as we can observe in Fig. 5. On

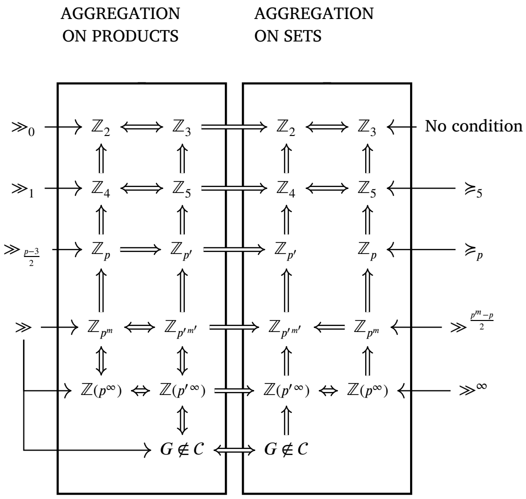


Fig. 5. Diagram of connections between the different groups regarding the preservation of T-subgroups, both on sets and on products for an arbitrary t-norm. The arrows $\Rightarrow$ indicate that any aggregation operator that preserves T-subgroups of the initial group also preserves T-subgroups of the final group. The arrows $\rightarrow$ assign the type of domination which characterizes the preservation of T-subgroups of the group that they are pointing to. p, p' are prime numbers such that $p \geq p'$ and m, m' are integers such that $m \geq m' \geq 1$.

the one hand, the existence of operators related by pseudo-p domination but not by type-k domination establishes a great difference between preservation of T-subgroups on sets and on products for those cyclic groups $\mathbb{Z}_p$ with $p \geq 5$. On the other hand, the existence of operators related by type-k asymmetric domination but not by domination ensures the difference between preservation of T-subgroups on sets and on products for those cyclic groups $\mathbb{Z}_{p^m}$ with $p^m > 4$ and $m > 1$ and for Prüfer's p-groups. Therefore, these families can be used to illustrate the great difference between aggregation on sets and aggregation on products.

It is important to note how this difference between the preservation of T-subgroups on sets and on products is translated to T-indistinguishability operators as it is shown in the following remark.

Remark 6.1. In their work [17], Formato, Gerla and Scarpati provided a method to obtain T-indistinguishablity operators from T-subgroups and vice versa. Recently, in [11], Boixader and Recasens studied this relation between T-subgroups and T-indistinguishability operators. A substantial amount of literature has been dedicated to the study of the aggregation of these indistinguishabilities. The conclusion obtained is that an aggregation operator preserves T-indistinguishability operators on sets if and only if it preserves T-indistinguishability operators on products if and only if the aggregation operator dominates the t-norm T (see [15,33]). Nevertheless, when we restrict ourselves to the class $\mathcal{TS}$ of T-indistinguishability operators that are obtained from T-subgroups, this last statement can be relaxed as a consequence of the results presented in this work. This implies that new aggregation operators can be employed to aggregate T-indistinguishabilities while maintaining their structure.

7. Conclusions and future work

In this paper, we have characterized the preservation of T-subgroups on sets when the lattice of the ambient group is a chain. We have introduced some new relations between an aggregation operator and a t-norm derived from domination, namely pseudo-p domination and type-k asymmetric domination. We have compared the relationships between these new concepts, as well as their connections to domination and type-k domination. We also conclude that, for any group G one of these four different forms of domination is a sufficient and necessary condition for an aggregation operator to preserve T-subgroups. The type of domination could be different depending on whether the aggregation is on sets or on products.

Therefore, this article completes the study started in [3,4,39] about the necessary and sufficient conditions for an aggregation operator to preserve T-subgroups. The combined results of these previous works together with the present one allows us to ensure that for T-subgroups defined over a given group, there is always a concrete relation that only depends on $\boldsymbol{A}$ and T, which holds when $\boldsymbol{A}$ preserves T-subgroups.

Moreover, new results on the aggregation of T-indistinguishability operators can be obtained as a consequence of this work. More precisely, the condition for an aggregation operator to preserve T-indistinguishability operators in the class $\mathcal{TS}$ can be relaxed as a consequence of the results presented in this article.

CRediT authorship contribution statement

Francisco Javier Talavera: Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization. **Sergio Ardanza-Trevijano:** Writing – review & editing, Validation, Investigation, Conceptualization. **Jean Bragard:** Writing – review & editing, Validation, Investigation, Conceptualization. **Jorge Elorza:** Writing – review & editing, Validation, Supervision, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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