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Aggregation of T-subgroups of groups whose subgroup lattice is a chain

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ABSTRACT

In this work, we study when an aggregation operator preserves the structure of T-subgroup of groups whose subgroup lattice is a chain. There are two widely used ways of defining the aggregation of structures in fuzzy logic, previously named *on sets* and *on products*. We will focus our attention on the one called aggregation *on products*. When the lattice of subgroups is not a chain, it is known that the dominance relation between the aggregation operator and the t-norm is crucial. We show that this property is again important for some of the groups in this study. However, for the rest of them, we must define a new property weaker than domination, that will allow us to characterize those operators which preserve T-subgroups.

1. Introduction

Information fusion in data handling and interpretation is essential in a wide variety of fields in science. Aggregation operators are functions that allow us to perform this fusion and have a strong presence in current research (see for instance [6,9,10,17]). When the data to be aggregated are endowed with a specific structure, it is of particular interest to know under which conditions this structure is preserved under fusion. In particular, the study of the preservation of fuzzy structures has been gaining relevance since its beginnings in [13,21] including structures such as T-indistinguishability operators or fuzzy implications (see, for instance, [12,14,23,25,27]). In all these studies, the preservation of structures under aggregation is studied according to two different definitions, but without highlighting this distinction. For example, Saminger, Mesiar and Bodenhofer proved in [27] that an aggregation operator A preserves T-indistinguishability operators if and only if A dominates the t-norm T. However, they did so with a definition of aggregation of T-indistinguishabilities different from the one later used by Drewniak and Dudziak in [12]. Although they obtained the same equivalence in that case, this is different if we look at other fuzzy structures. In 2021, Pedraza, Rodriguez-López and Valero (see [22]) fixed the aggregation notation *on sets* and *on products* to be able to distinguish both definitions. They also showed that these definitions are not equivalent in the context of preservation of fuzzy quasi-metrics among others. A more exhaustive distinction between those works using either definitions of aggregation of fuzzy structures can be found in [3].

Since its definition by Rosenfeld in [26], a large amount of works regarding fuzzy subgroups have been developed (see [1,11,20,28]). In this context, there exists a close relation between T-subgroups and T-indistinguishability operators (see [8,15]). More precisely, we can define T-indistinguishability operators making use of T-subgroups and vice versa. This relation provides several

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fields of possible application of T-subgroups such us image processing, fuzzy classification systems under uncertainty, imprecision or vagueness, or approximate reasoning (see, for instance, [7,24,29]).

The study of the preservation of T-subgroups started in [4], where a characterization of the aggregation operators that preserve T_M-subgroups on sets was given. Note that T_M denotes the minimum t-norm. Some of the authors of the present contribution continued in the same direction with a more general study (see [3]), which analyzed the relationships between the two definitions of aggregation of T-subgroups. In these papers, they had to distinguish groups according to their lattice of subgroups. We denote the lattice of subgroups of a group G by $Lat(G)$ and it is said that this lattice is a chain if for all $H_1, H_2 \in G$ subgroups of G, either $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$. $\mathcal{C}$ shall be the set of all groups whose lattice of subgroups is a chain, i.e.:

$$\mathcal{C} = \{G \mid G \text{ is a group and } Lat(G) \text{ is a chain}\}.$$

The authors of [3] found necessary and sufficient conditions for an aggregation operator to preserve T-subgroups on both *products* and *sets* when we have an ambient group $G \notin \mathcal{C}$. Moreover, they showed which aggregation operators preserve T_M-subgroups when $G \in \mathcal{C}$. However, the study of aggregation operators that preserve T-subgroups for arbitrary t-norms and $G \in \mathcal{C}$ has not yet been considered. In the context of the aforementioned applications, it is interesting to study the case of t-norms apart from the minimum to extend the possibilities for practitioners.

Section 2 establishes some important definitions that will be used throughout the article. Section 3 presents some background results that will be very useful in the sequent proofs and discussion. Moreover, we present some new important definitions. In Section 4, we study the necessary and sufficient conditions for the preservation of T-subgroups *on products* when the ambient group $G \in \mathcal{C}$, and we provide further implications *on sets*. Finally, Section 5 gives a general overview and establishes some open problems.

2. Preliminaries

In this section we present some general results and definitions that will be useful throughout the paper.

Definition 2.1 *([19])*. A triangular norm, t-norm for short, is a binary operation $T : [0,1]^2 \to [0,1]$ such that for all $x, y, z \in [0,1]$ the following axioms are satisfied:

$T1$. $T(x,y) = T(y,x)$. (Commutativity)
$T2$. $T(x, T(y,z)) = T(T(x,y), z)$. (Associativity)
$T3$. $T(x,y) \leq T(x,z)$ whenever $y \leq z$. (Monotonicity)
$T4$. $T(x,1) = x$. (Boundary condition)

Example 2.2. Here we present some important examples of t-norms:

1. $T_M(x,y) = \min\{x,y\}$. (Minimum t-norm)
2. $T_D(x,y) = \begin{cases} 0 & \text{if } \max\{x,y\} < 1, \\ \min\{x,y\} & \text{if } \max\{x,y\} = 1. \end{cases}$ (Drastic t-norm)
3. $T_P(x,y) = xy$. (Product t-norm)
4. $T_L(x,y) = \max\{x+y-1, 0\}$. (Łukasiewicz t-norm)
5. $T_{P/s}(x,y) = \begin{cases} \dfrac{xy}{s} & \text{if } \max\{x,y\} < 1 \\ \min\{x,y\} & \text{if } \max\{x,y\} = 1 \end{cases}$ with $s \in [1, \infty)$.

Remark 2.3. The family of t-norms $T_{P/s}$ from the previous example satisfies:

$$T_D \leq T_{P/s_2} \leq T_{P/s_1} \leq T_P \text{ for } 1 \leq s_1 < s_2 < \infty.$$

These t-norms are going to be instrumental in the obtention of some counterexamples in this manuscript.

Fuzzy subgroups were first defined by Rosenfeld in [26]. In [1,2], Anthony and Sherwood redefined them to include an arbitrary t-norm T and a normalization condition. Let X be a non-empty set and let $[0,1]^X$ denote the set of all fuzzy sets $\mu : X \to [0,1]$.

Definition 2.4. Let G be a group, $\mu \in [0,1]^G$ a fuzzy subset of G and T a t-norm. μ is called fuzzy T-subgroup of G if:

G1. $\mu(e) = 1$ where $e \in G$ denotes the neutral element.
G2. $\mu(x) = \mu(x^{-1})$ $\forall x \in G$.
G3. $\mu(xy) \geq T(\mu(x), \mu(y))$ $\forall x, y \in G$.

Definition 2.5 *([26])*. Let G be a group and μ a fuzzy set of G. For each $t \in [0,1]$, the level set μ_t and the strict level set μ^t are defined as follows:

$$\mu_t = \{x \in G \mid \mu(x) \geq t\}; \qquad \mu^t = \{x \in G \mid \mu(x) > t\}.$$

The support of μ is defined by supp $\mu = \mu^0$.

Proposition 2.6 *([11]). Let G be a group and μ a fuzzy set of G, then μ is a T_M-subgroup of G if and only if all its non-empty level sets are subgroups of G and $\mu(e) = 1$.*

Definition 2.7 *([9]).* Let $\boldsymbol{A} : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ be a function. $\boldsymbol{A}$ is called aggregation operator or aggregation function if:

A1. For all $\boldsymbol{x} = (x_1, \dots, x_n)$, $\boldsymbol{y} = (y_1, \dots, y_n) \in [0,1]^n$ such that $x_i \leq y_i$ $\forall i \in \{1, \dots, n\}$, we have $\boldsymbol{A}(x_1, \dots, x_n) \leq \boldsymbol{A}(y_1, \dots, y_n)$.
A2. $\boldsymbol{A}(\mathbf{0}) = \boldsymbol{A}(0, \dots, 0) = 0$ and $\boldsymbol{A}(\mathbf{1}) = A(1, \dots, 1) = 1$.
A3. $\boldsymbol{A}(x) = x$ for all $x \in [0,1]$.

Each aggregation operator $\boldsymbol{A}$ can be represented as a collection of n-ary aggregation operators $A_{(n)} : [0,1]^n \to [0,1]$ satisfying A1 and A2 if $n \geq 2$ and additionally A3 if $n = 1$. In order to shorten some of the proofs, we will use the notation A instead of $A_{(n)}$ if the context is clear enough.

We can follow [3] to define the aggregation of fuzzy T-subgroups, both on products and on sets.

Definition 2.8 *([3]).* Let $\boldsymbol{A} : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$, be an aggregation operator, T a t-norm and $n \in \mathbb{N}$. Given n fuzzy T-subgroups $\mu_1, \dots, \mu_n$ of a group G, we denote by $\boldsymbol{\mu}$ the map $\boldsymbol{\mu} : G \to [0,1]^n$, with:

$$\boldsymbol{\mu}(x) = (\mu_1(x), \dots, \mu_n(x))$$

for x in G and we define the **aggregation of fuzzy subgroups on sets** as $A \circ \boldsymbol{\mu}$, where:

$$A \circ \boldsymbol{\mu}(x) = A(\mu_1(x), \dots, \mu_n(x)).$$

We will say that **A preserves the structure of T-subgroup on sets** if and only if $A \circ \boldsymbol{\mu}$ is a T-subgroup for any $\boldsymbol{\mu}$ as above.

In the same way $\tilde{\boldsymbol{\mu}}$ denotes $\tilde{\boldsymbol{\mu}} : \prod_{i=1}^n G \to [0,1]^n$ with:

$$\tilde{\boldsymbol{\mu}}(\boldsymbol{x}) = (\mu_1(x_1), \dots, \mu_n(x_n))$$

for $\boldsymbol{x} = (x_1, \dots, x_n)$ in $\prod_{i=1}^n G$. We define the **aggregation of fuzzy subgroups on products** as $A \circ \tilde{\boldsymbol{\mu}}$, where:

$$A \circ \tilde{\boldsymbol{\mu}}(\boldsymbol{x}) = A(\mu_1(x_1), \dots, \mu_n(x_n)).$$

As before, we state that **A preserves the structure of T-subgroup on products** if and only if $A \circ \tilde{\boldsymbol{\mu}}$ is a T-subgroup for any $\tilde{\boldsymbol{\mu}}$ as above.

3. Aggregation of T-subgroups and domination

In this section we compile some known results on aggregation of T-subgroups that will be useful throughout this paper. We also introduce the dominance relation and we define new weaker types of domination that are going to be determinant in the following.

Proposition 3.1 *([3]). Let G be a non-trivial group, $\boldsymbol{A} : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ an aggregation operator, and $\mu_1, \dots, \mu_n$ T-subgroups, then the fuzzy subgroup axioms $G1$ and $G2$ are satisfied by $A \circ \boldsymbol{\mu}$ and $A \circ \tilde{\boldsymbol{\mu}}$.*

Hence, in order to characterize the situations where $\boldsymbol{A}$ preserves T-subgroups of G, it is enough to prove whether or not an aggregation operator preserves the fuzzy subgroup axiom $G3$.

Proposition 3.2 in [3] shows that the preservation of T-subgroups on products implies the preservation of T-subgroups on sets.

Proposition 3.2 *([3]). Let G be a non trivial group, T a t-norm, and $\boldsymbol{A}$ an aggregation function. If $\boldsymbol{A}$ preserves the structure of T-subgroup on products then $\boldsymbol{A}$ also preserves this structure on sets.*

The dominance relation introduced in [27] plays a key role in the preservation of different fuzzy properties as is the case of T-indistinguishability operators or T-subgroups of a group $G \notin C$ (see [3,22,27]).

Definition 3.3 *([27]).* Consider an n-ary aggregation operator $A_{(n)}$ and an m-ary aggregation operator $B_{(m)}$. We say that $A_{(n)}$ dominates $B_{(m)}$ if for all $x_{i,j} \in [0,1]$ with $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$, the following property holds:

$$B_{(m)}(A_{(n)}(x_{1,1}, \dots, x_{1,n}), \dots, A_{(n)}(x_{m,1}, \dots, x_{m,n})) \\ \leq A_{(n)}(B_{(m)}(x_{1,1}, \dots, x_{m,1}), \dots, B_{(m)}(x_{1,n}, \dots, x_{m,n})).$$

Let now $\boldsymbol{A}$ and $\boldsymbol{B}$ be aggregation operators. We say that $\boldsymbol{A}$ dominates $\boldsymbol{B}$ if $A_{(n)}$ dominates $B_{(m)}$ for all $n, m \in \mathbb{N}$.

Remark 3.4. Given that t-norms are associative aggregation operators, $\boldsymbol{A}$ dominates T if and only if $A_{(n)}$ dominates T for all integers $n > 1$. That is, for each n:

$$A(T(x_1, y_1), \dots, T(x_n, y_n)) \geq T(A(x_1, \dots, x_n), A(y_1, \dots, y_n))$$

for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in [0,1]^n$.

The domination property is quite strong and it always ensures the preservation of T-subgroups as we can see in [3] (Theorem 3.14).

Theorem 3.5 *([3]). Let G be a group and T a t-norm. If $\boldsymbol{A}$ is an aggregation operator that dominates T, then $\boldsymbol{A}$ preserves the structure of T-subgroup on products and hence on sets.*

When the subgroup lattice of the ambient group is not a chain, the aggregation operators preserving the structure of T-subgroup on sets are the same that the ones preserving this structure on products. Moreover, the only operators to do so are those which dominate the t-norm. This is a consequence of Theorem 3.18 and Corollary 3.19 in [3].

Theorem 3.6 *([3]). Let be a group $G \notin C$, T a t-norm and $\boldsymbol{A} : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ an aggregation operator. The following are equivalent:*

(i) $\boldsymbol{A}$ dominates T.
(ii) $\boldsymbol{A}$ preserves the structure of T-subgroup on sets.
(iii) $\boldsymbol{A}$ preserves the structure of T-subgroup on products.

Note that in the previous theorem we have imposed $G \notin C$ in order to obtain the converse implication of Theorem 3.5 for the aggregation of T-subgroups on sets and on products.

From now on, we will focus on the study of preservation of T-subgroups when $G \in C$. Theorems 3.5 and 3.10 in [3] show what happens in case of $T = T_M$. They can be stated as follows.

Theorem 3.7 *([3]). Let G be a non-trivial group. The following are equivalent:*

(i) $G \in C$.
(ii) Every aggregation function preserves the structure of T_M-subgroup on sets.

Theorem 3.8 *([3]). Let G be a group and $\boldsymbol{A}$ an aggregation function. Then the following are equivalent:*

(i) $\boldsymbol{A}$ dominates the minimum t-norm, T_M.
(ii) $\boldsymbol{A}$ preserves the structure of T_M-subgroup on products.

Remark 3.9. Note that all proofs in [3] include both finite and infinite groups. For the subsequent results, the two possibilities will have to be considered separately. The case of infinite groups is studied in subsection 4.5.

Thus, the preservation of T-subgroups, both on sets and on products when $T \neq T_M$ and $G \in C$ was not addressed in previous studies and will be the subject of the next section. However, we must first define some new concepts related to domination which play a central role in this work. In order to do that, we recall some notation and properties regarding t-norms.

Definition 3.10 *([18]).* Let T be a t-norm, $x \in [0,1]$ and $k \in \mathbb{N}$. We define $x_T^{(k)}$ as:

$$x_T^{(k)} = \begin{cases} x & \text{if } k = 1, \\ T(x_T^{(k-1)}, x) & \text{if } k \neq 1. \end{cases}$$

Remark 3.11. Note that, in the sense of Definition 3.10, the sequence $\{x_T^{(k)}\}_{k\in\mathbb{N}}$ is non-increasing due to the monotonicity of the t-norm T.

Lemma 3.12 *([5]). Let μ be a T-subgroup of a group G and $k \in \mathbb{N}$. Then:*

$$\mu(a^k) \geq \mu(a)_T^{(k)} \; \forall a \in G.$$

Lemma 3.13 *([5]). Given a t-norm T and a number $k \in \mathbb{N}$ greater than or equal to 2, we have:*

$$x_T^{(k)} = T(x_T^{(k_1)}, x_T^{(k_2)})$$

for all $x \in [0,1]$ and $k_1, k_2 \in \mathbb{N}$ such that $k_1 + k_2 = k$.

The next definition suggests weaker forms of domination.

Definition 3.14 *(Type-k domination).* Given $k \in \mathbb{N}$, we say that an aggregation operator $\boldsymbol{A}$, type-k dominates a t-norm T if:

$$A(T(x_1, y_1), \dots, T(x_n, y_n)) \geq T(A(x_1, \dots, x_n), A(y_1, \dots, y_n)), \tag{1}$$

for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in [0,1]^n$ such that for each $i \in \{1, \dots, n\}$ one of the following conditions applies:

- $\max\{x_i, y_i\} = 1$.
- $\min\{x_i, y_i\} \geq (\max\{x_i, y_i\})_T^{(k)}$.

We denote this fact by $\boldsymbol{A} \gg_k T$.

For $k = 0$ we say that $\boldsymbol{A}$ type-0 dominates T if (1) holds for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in [0,1]^n$ such that $\max\{x_i, y_i\} = 1$ for each $i \in \{1, \dots, n\}$ and we denote it $\boldsymbol{A} \gg_0 T$.

Remark 3.15. Let us list some of the features of type-k domination:

1. If we set $k = 1$, the statement $\min\{x, y\} \geq (\max\{x, y\})_T^{(k)}$ is equivalent to the condition $x = y$ for any $x, y \in [0,1]$. Therefore, $\boldsymbol{A} \gg_1 T$ if and only if the inequality (1) holds for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in [0,1]^n$ such that for each $i \in \{1, \dots, n\}$ either $\max\{x_i, y_i\} = 1$ or $x_i = y_i$ (Fig. 1b).
2. Note that type-k domination relaxes the conditions of domination since it reduces the domain where inequality (1) is required. This domain restriction only depends on the t-norm and the value of k. Moreover:

   $$(\max\{x, y\})_T^{(k)} \geq (\max\{x, y\})_T^{(k+1)} \geq 0$$

   for all $k \geq 1$. This suggests that there are going to be more points satisfying the inequality $\min\{x, y\} \geq (\max\{x, y\})_T^{(k+1)}$ than $\min\{x, y\} \geq (\max\{x, y\})_T^{(k)}$. Hence, the property $\boldsymbol{A} \gg_{k+1} T$ is more restrictive than $\boldsymbol{A} \gg_k T$ and, therefore, closer to the property of domination in that respect. As a consequence, given $k_1, k_2 \in \mathbb{N}$ where $k_1 \geq k_2$, we have the following chain of implications:

   $$\boldsymbol{A} \gg T \Rightarrow \boldsymbol{A} \gg_{k_1} T \Rightarrow \boldsymbol{A} \gg_{k_2} T \Rightarrow \boldsymbol{A} \gg_0 T.$$

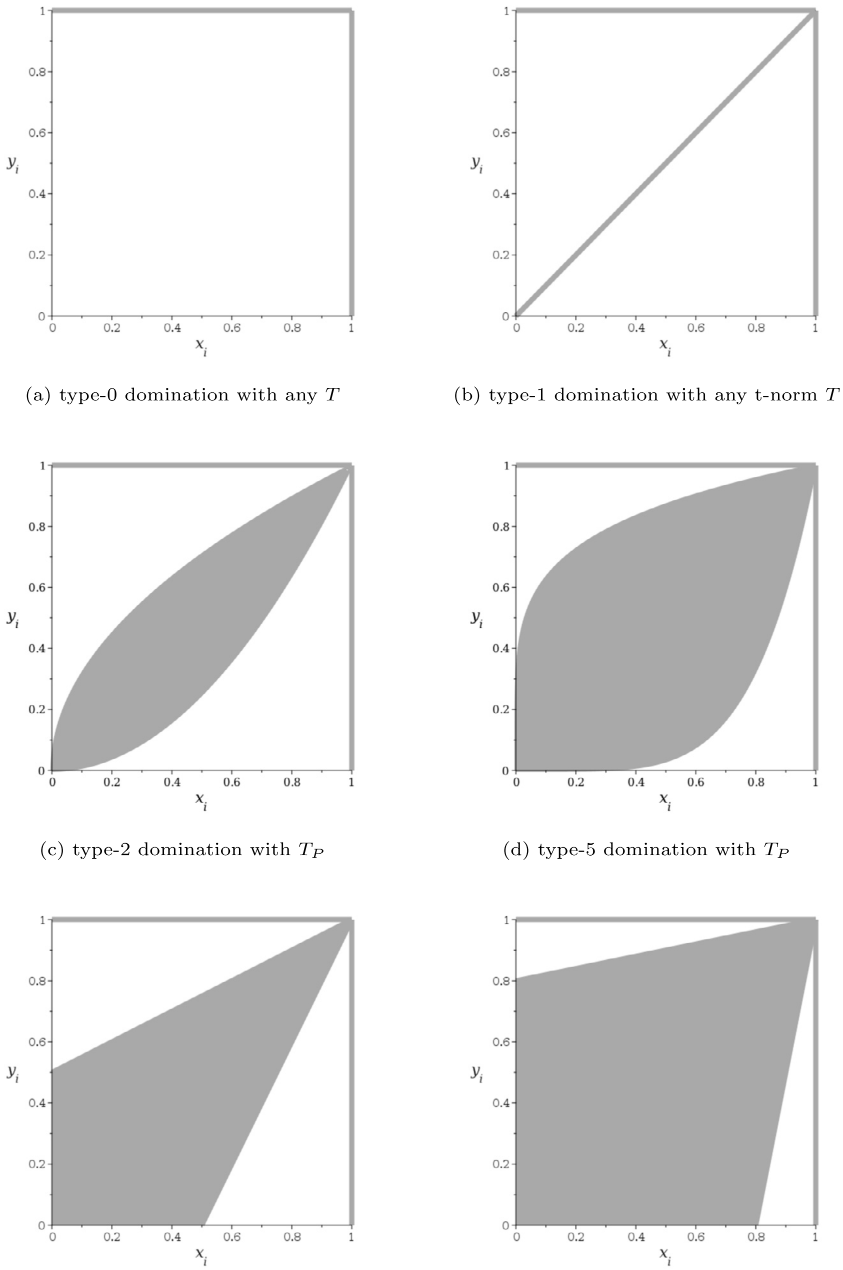
   Domination is then stronger than type-k domination for all $k \in \mathbb{N}$ and this in turn, is stronger than type-0 domination. We will see in later examples that these dominations are different in general.
3. In Fig. 1 we show for which points inequality (1) should be imposed so that $\boldsymbol{A} \gg_k T$ for some chosen k and different t-norms. This inequality should be true whenever any point (x_i, y_i) lies in the shaded area for all $i \in \{1, \dots, n\}$. Note that these areas do not depend on the selected t-norm when $k = 0$ (Fig. 1a) or $k = 1$ (Fig. 1b). However, taking T_P and T_L we can check that this is not true for greater values of k (Figs. 1c, 1d, 1e and 1f). Hence when defining type-k domination we are restricting the domain where inequality (1) holds. This restriction only depends on the t-norm.

Type-k domination will play a fundamental role in the preservation of T-subgroups on products when $G \in \mathcal{C}$.

4. Aggregation *on products* when the subgroup lattice of the ambient group is a chain

The aim of this section is to analyze the preservation of T-subgroups on products when T is an arbitrary t-norm. That is, to find a result analogous to Theorem 3.8 without imposing the t-norm to be the minimum. The dominance property will again play a crucial role since, in any case, it is a sufficient condition for an aggregation operator to preserve T-subgroups on products (hence also on sets). This fact allows us to claim that those aggregation operators that preserve T-subgroups of $G \in \mathcal{C}$ must be related to T by means of a property equal to or less restrictive than domination. Indeed, when domination is too strong, type-k domination will be the appropriate condition to ensure the preservation of the T-subgroup structure. All the main results of this article are intended to elucidate the nature of these relations.

First of all, we must bear in mind that the only finite groups where $Lat(G)$ is a chain are cyclic. Moreover, those groups have p^m elements with $m \in \mathbb{N}$ and p a prime number. The only infinite groups in $\mathcal{C}$ are the Prüfer p-groups $\mathbb{Z}(p^\infty)$ where p is a prime number. There are different ways of defining these groups. Prüfer groups can be defined as the set of the p^mth complex roots of unity for all $m \in \mathbb{N}$ with the complex multiplication as the operation. It may also be defined as $\mathbb{Q}^{(p)}/\mathbb{Z}$ where $\mathbb{Q}^{(p)}$ is the group of all rational



(a) type-0 domination with any T

(b) type-1 domination with any t-norm T

(c) type-2 domination with T_P

(d) type-5 domination with T_P

(e) type-2 domination with T_L

(f) type-5 domination with T_L

Fig. 1. To ensure that $\boldsymbol{A} \gg_k T$, (1) must hold as long as the point (x_i, y_i) lies in the gray area of the corresponding plot for all $i \in \{1, \ldots, n\}$.

numbers whose denominators are powers of p. Therefore, all proper subgroups of $\mathbb{Z}(p^\infty)$ are cyclic, of finite order and isomorphic to $\mathbb{Z}_{p^m}$ for some $m \in \mathbb{N}$. A detailed study of Prüfer groups and their properties can be found in [16].

Once we have established which groups are included in $\mathcal{C}$, we will carry out a systematic study depending on the properties of these groups, since the relations between $\boldsymbol{A}$ and T that characterize the preservation of T-subgroups on behalf of $\boldsymbol{A}$ will depend on the ambient group. To begin with, we prove in Example 4.1 that, in general, not every operator preserves T-subgroups on products.

Example 4.1. Let G be a group. Define the function $\boldsymbol{A} : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ so that its n-ary operators have the form:

$$A(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if } x_i = 1 \text{ for some } i \in \{1, \ldots, n\}, \\ 0 & \text{otherwise .} \end{cases}$$

It is easy to prove that $\boldsymbol{A}$ is an aggregation operator. Let us take now $x_1, \dots, x_n \in]0,1[$ and define the fuzzy sets $\mu_1, \dots, \mu_n$ such that for each $i \in \{1, \dots, n\}$:

$$\mu_i(z) = \begin{cases} 1 & \text{if } z = e, \\ x_i & \text{if } z \neq e. \end{cases}$$

Since the level sets of each of these fuzzy sets are subgroups of G, we have that $\mu_1, \dots, \mu_n$ are T_M-subgroups by Proposition 2.6 and therefore they are also T-subgroups. Let us choose:

$$\boldsymbol{a} = (a_1, \dots, a_n), \boldsymbol{b} = (b_1, \dots, b_n) \in G^n$$

with $a_1, b_2, \dots, b_n \in G \setminus \{e\}$ and $b_1 = a_2 = \dots = a_n = e$. Here, $a_i b_i \neq e$ for all $i \in \{1, \dots, n\}$ and:

$$\begin{aligned} A \circ \tilde{\boldsymbol{\mu}}(\boldsymbol{ab}) = A(x_1, \dots, x_n) = 0 < 1 &= T(A(x_1, 1, \dots, 1), A(1, x_2, \dots, x_n)) \\ &= T(A \circ \tilde{\boldsymbol{\mu}}(\boldsymbol{a}), A \circ \tilde{\boldsymbol{\mu}}(\boldsymbol{b})). \end{aligned}$$

Consequently, $\boldsymbol{A}$ is an aggregation operator that does not preserve T-subgroups on products.

In addition, the following example shows that domination is not a necessary condition in general for the preservation of T-subgroups when $G \in \mathcal{C}$. Theorem 3.8 does not hold then when we employ some t-norms other than the minimum.

Example 4.2. Let $\boldsymbol{A} = T_P$ and $T = T_{P/2}$. In [3] we found that $\boldsymbol{A} \not\gg T$. Nevertheless $\boldsymbol{A}$ actually preserves T-subgroups of $G = \mathbb{Z}_2$. Given $\mu_1, \dots, \mu_n$ T-subgroups of G, it was verified that for all $\boldsymbol{a}, \boldsymbol{b} \in G^n$:

$$\begin{aligned} T_P \circ \tilde{\boldsymbol{\mu}}(\boldsymbol{ab}) = \prod_{i=1}^{n} \mu_i(a_i b_i) &\geq T_{P/2}\left(\prod_{i=1}^{n} \mu_i(a_i), \prod_{i=1}^{n} \mu_i(b_i)\right) \\ &= T_{P/2}(T_P \circ \tilde{\boldsymbol{\mu}}(\boldsymbol{a}), T_P \circ \tilde{\boldsymbol{\mu}}(\boldsymbol{b})). \end{aligned} \tag{2}$$

Thus, domination is not a necessary condition for the preservation of T-subgroups when $T \neq T_M$ although it is a sufficient condition.

In order to characterize the aggregation operators preserving T-subgroups on products when $Lat(G)$ is a chain, we will make use of type-k domination whenever full domination is not necessary. Depending on the order of the group of $\mathcal{C}$ under consideration, we will need to take a different value of k. We detail the particular case treated on each subsection:

4.1 Groups with 2 and 3 elements.
4.2 Cyclic groups with 4 and 5 elements.
4.3 Groups with prime order greater than 5.
4.4 Cyclic groups with order $p^m > 4$, being p a prime and m an integer greater than or equal to 2.
4.5 Infinite Prüfer p-groups.

The following result establishes that any aggregation operator that preserves T-subgroups on products must type-0 dominate T.

Theorem 4.3. *Let G be a group, T a t-norm and $\boldsymbol{A}$ an aggregation operator such that $\boldsymbol{A}$ preserves T-subgroups of G on products. Then $\boldsymbol{A} \gg_0 T$.*

Proof. Fixing $(x_1, \dots, x_n), (y_1, \dots, y_n) \in [0,1]^n$ such that $\max\{x_i, y_i\} = 1$ for all $i \in \{1, \dots, n\}$, we can define $\mu_1, \dots, \mu_n$ T_M-subgroups of G as follows:

$$\mu_i(z) = \begin{cases} 1 & \text{if } z = e, \\ \min\{x_i, y_i\} & \text{if } z \neq e \end{cases} \quad \text{for each } i \in \{1, \dots, n\}.$$

They are in fact T_M-subgroups due to Proposition 2.6. Let us take $\boldsymbol{a} = (a_1, \dots, a_n), \boldsymbol{b} = (b_1, \dots, b_n) \in G^n$ such that for each $i \in \{1, \dots, n\}$:

- $a_i \neq e$, $b_i = e$ if $\min\{x_i, y_i\} \in \{x_i, 1\}$,
- $a_i = e$, $b_i \neq e$ if $\min\{x_i, y_i\} = y_i < 1$.

In all cases we have that $x_i = \mu_i(a_i)$ and $y_i = \mu_i(b_i)$. Moreover $a_i b_i \neq e$ and $\max\{x_i, y_i\} = 1$ for all $i \in \{1, \dots, n\}$, so $\mu_i(a_i b_i) = \min\{x_i, y_i\} = T(x_i, y_i)$.

Since we have assumed that $\boldsymbol{A}$ preserves T-subgroups of G on products, then:

$$\begin{aligned} A(T(x_1, y_1), \dots, T(x_n, y_n)) &= A \circ \tilde{\boldsymbol{\mu}}(a_1 b_1, \dots, a_n b_n) = A \circ \tilde{\boldsymbol{\mu}}(\boldsymbol{ab}) \\ &\geq T(A \circ \tilde{\boldsymbol{\mu}}(\boldsymbol{a}), A \circ \tilde{\boldsymbol{\mu}}(\boldsymbol{b})) \end{aligned}$$

$$= T(A(\mu_1(a_1), \dots, \mu_n(a_n)), A(\mu_1(b_1), \dots, \mu_n(b_n)))$$
$$= T(A(x_1, \dots, x_n), A(y_1, \dots, y_n)).$$

Therefore, $\boldsymbol{A} \gg_0 T$. □

Remark 4.4. This theorem together with Theorem 3.5 states that, for any group, any property that characterizes the preservation of T-subgroups on products under aggregation must be weaker than domination but stronger than type-0 domination. We will see that such property is type-k domination.

4.1. Groups with 2 and 3 elements

We will show now that, if the cardinal of the group G is 2 or 3, the converse of Theorem 4.3 is also true. However, this situation does not hold true in general for other groups as we will see in the next section.

Theorem 4.5. *Let T be a t-norm, $\boldsymbol{A}$ an aggregation operator and G a group with 2 or 3 elements. The following statements are equivalent:*

(i) $\boldsymbol{A}$ preserves T-subgroups of G on products.
(ii) $\boldsymbol{A} \gg_0 T$.

Proof. $(i) \Rightarrow (ii)$ It is a direct consequence of Theorem 4.3.
$(ii) \Rightarrow (i)$ Let us suppose that $\boldsymbol{A} \gg_0 T$. Given $n \in \mathbb{N}$, $\mu_1, \dots, \mu_n$ T-subgroups of G and $\boldsymbol{a}, \boldsymbol{b} \in G^n$, let us check that:

$$A \circ \tilde{\boldsymbol{\mu}}(\boldsymbol{ab}) \geq T(A \circ \tilde{\boldsymbol{\mu}}(\boldsymbol{a}), A \circ \tilde{\boldsymbol{\mu}}(\boldsymbol{b})).$$

Since $|G| \in \{2, 3\}$ and $G1$ and $G2$ must be satisfied, all T-subgroups must be in the form:

$$\mu_i(z) = \begin{cases} 1 & \text{if } z = e, \\ \alpha_i & \text{if } z \neq e \end{cases}$$

with $\alpha_i \in [0, 1]$ for each $i \in \{1, \dots, n\}$. We will choose $(x_1, \dots, x_n), (y_1, \dots, y_n) \in [0, 1]^n$ such that for every $i \in \{1, \dots, n\}$:

$$x_i \geq \mu_i(a_i), \quad y_i \geq \mu_i(b_i), \quad \mu_i(a_i b_i) \geq T(x_i, y_i) \quad \text{and} \quad \max\{x_i, y_i\} = 1. \tag{3}$$

With this constrains, the points $(x_1, \dots, x_n), (y_1, \dots, y_n)$ satisfy inequality (1) in Definition 3.14. As we supposed that $\boldsymbol{A} \gg_0 T$:

$$\begin{aligned} A \circ \tilde{\boldsymbol{\mu}}(\boldsymbol{ab}) &= A(\mu_1(a_1 b_1), \dots, \mu_1(a_n b_n)) \geq A(T(x_1, y_1), \dots, T(x_n, y_n)) \\ &\geq T(A(x_1, \dots, x_n), A(y_1, \dots, y_n)) \\ &\geq T(A(\mu_1(a_1), \dots, \mu_n(a_n)), A(\mu_1(b_1), \dots, \mu_n(b_n))) \\ &= T(A \circ \tilde{\boldsymbol{\mu}}(\boldsymbol{a}), A \circ \tilde{\boldsymbol{\mu}}(\boldsymbol{b})). \end{aligned} \tag{4}$$

In order to choose each pair (x_i, y_i), we will proceed as follows. If $\mu_i(a_i b_i) = 1$, it suffices to fix $(x_i, y_i) = (1, 1)$. So let us suppose that $\mu_i(a_i b_i) \neq 1$. In that case, and given that μ_i is T_M-subgroup, we have that $1 > \alpha_i = \mu_i(a_i b_i) = \min\{\mu_i(a_i), \mu_i(b_i)\}$. There are two possibilities here:

- If $\mu_i(a_i) = \min\{\mu_i(a_i), \mu_i(b_i)\}$, we will take $x_i = \mu_i(a_i) = \mu_i(a_i b_i)$ and $y_i = 1 \geq \mu_i(b_i)$.
- If $\mu_i(b_i) = \min\{\mu_i(a_i), \mu_i(b_i)\}$, we will take $x_i = 1 \geq \mu_i(a_i)$ and $y_i = \mu_i(b_i) = \mu_i(a_i b_i)$.

Hence, (3) and (4) holds and $\boldsymbol{A}$ preserves T-subgroups. □

Under the conditions of the above theorem we obtain the next corollary as a consequence of Proposition 3.2.

Corollary 4.6. *Let T be a t-norm, $\boldsymbol{A}$ an aggregation operator, and G a group with 2 or 3 elements. If $\boldsymbol{A} \gg_0 T$ then $\boldsymbol{A}$ preserves T-subgroups on sets.*

Remark 4.7. Note that, when $T = T_M$ Theorem 3.8 states that $\boldsymbol{A} \gg T_M$ if and only if $\boldsymbol{A}$ preserves T_M-subgroups on products. If $|G| \in \{2, 3\}$, Theorem 4.5 establishes that $\boldsymbol{A}$ preserves T-subgroups if and only if $\boldsymbol{A} \gg_0 T$. Thus, we can draw the scheme of implications of the Fig. 2 and claim that $\boldsymbol{A} \gg T_M$ if and only if $\boldsymbol{A} \gg_0 T_M$.

Since both properties do not depend on the chosen ambient group, the latter equivalence can be proved directly by means of the following characterization of dominance over T_M provided by Saminger, Mesiar and Bodenhofer in [27]:

$$\begin{tikzcd} \boldsymbol{A} \gg T \arrow[r, Rightarrow] & \boldsymbol{A} \text{ preserves } T\text{-subgroups on products} \arrow[r, Rightarrow] \arrow[l, Rightarrow, bend left, "\text{if } T = T_M"] & \boldsymbol{A} \gg_0 T \arrow[l, Rightarrow, bend left, "\text{if } |G| \in \{2,3\}"] \end{tikzcd}$$

Fig. 2. Relation between domination, type-0 domination and preservation of T-subgroups on products.

$$\boldsymbol{A} \gg T_M \text{ if and only if}$$

$$A(x_1,\dots,x_n) = \min\{A(x_1,1,\dots,1),\dots,A(1,\dots,1,x_n)\}$$

for all $(x_1,\dots,x_n),(y_1,\dots,y_n) \in [0,1]^n$.

Nevertheless, type-0 domination and usual domination are not equivalent in general. Example 4.2 shows that $T_P \not\gg T_{P/2}$ but T_P preserves $T_{P/2}$-subgroups on products. By Theorem 4.5 we have that $T_P \gg_0 T_{P/2}$ so this example highlights the differences between both forms of domination, $\gg_0$ and $\gg$.

4.2. Cyclic groups with 4 and 5 elements

Let us illustrate with an example that type-0 domination is not a sufficient condition for the preservation of T-subgroups on products if the ambient group is cyclic of order greater than 4.

Example 4.8. In Remark 4.7 we showed that $T_P \gg_0 T_{P/2}$. Let us see that, if $G = \langle g \rangle$ with $|G| = r \geq 4$, we can construct $T_{P/2}$-subgroups such that their aggregation under T_P is not a $T_{P/2}$-subgroup. Let n be a natural number greater than 2 and the T-subgroups:

$$\mu_i(z) = \begin{cases} 1 & \text{if } z = e, \\ \frac{1}{2} & \text{if } z \in \{g, g^{r-1}\}, \\ \frac{1}{7} & \text{otherwise} \end{cases}$$

for each $i \in \{1,\dots,n\}$. It is easy to prove that these are indeed T-subgroups of any of the aforementioned groups. In addition, if we fix $\boldsymbol{a},\boldsymbol{b} \in G^n$ so that $a_i = b_i = g$ for all $i \in \{1,\dots,n\}$, we obtain that $\mu_i(a_i) = \mu(b_i) = \frac{1}{2}$ and $\mu(a_i b_i) = \mu(g^2) = \frac{1}{7}$. Thus:

$$T_P \circ \tilde{\boldsymbol{\mu}}(\boldsymbol{ab}) = \frac{1}{7^n} < \frac{1}{2^{2n+1}} = T_{P/2}\left(\frac{1}{2^n}, \frac{1}{2^n}\right) = T_{P/2}(T_P \circ \tilde{\boldsymbol{\mu}}(\boldsymbol{a}), T_P \circ \tilde{\boldsymbol{\mu}}(\boldsymbol{b})),$$

concluding that there is no preservation of $T_{P/2}$-subgroups.

Before looking for the relationship between $\boldsymbol{A}$ and T to guarantee the preservation of T-subgroups in these new cases, we need to introduce some results that provide information on the structure of T-subgroups in cyclic groups.

Lemma 4.9. *Let μ be a T-subgroup of a cyclic group G with order r. Then we have that either $\mu(a) = 1$ for all $a \in G$ or $\mu(b) \neq 1$ for every b generator of the group.*

Proof. Given $b \in G$ such that $G = \langle b \rangle$. If $\mu(b) = 1$ we can take $a \in G \setminus \{e\}$ so that there exists $k \in \{1,\dots,r-1\}$ with $a = b^k$. Hence:

$$1 \geq \mu(a) = \mu(b^k) \geq \mu(b)_T^{(k)} = 1_T^{(k)} = 1$$

and then $\mu(a) = 1$. Thus, it follows that if $\mu(b) \neq 1$ for some generator b, then μ does not take the value 1 for any other generator. □

Corollary 4.10. *Let μ be a T-subgroup of a group G with a prime number p of elements. Then we have that either $\mu(a) = 1$ for all $a \in G$ or $\mu(a) \neq 1$ for all $a \in G \setminus \{e\}$.*

With the following theorem, we get to characterize the T-subgroups of a cyclic group.

Theorem 4.11. *Let $\mu \in [0,1]^G$ with $G = \langle g \rangle$ a cyclic group with order $r \in \mathbb{N}$. The following are equivalent:*

(i) *μ is a T-subgroup of G.*
(ii) *μ must be of the form:*

$$\mu(z) = \begin{cases} 1 & \text{if } z = e, \\ \alpha_1 & \text{if } z \in \{g, g^{-1}\}, \\ \vdots & \qquad \vdots \\ \alpha_{\left[\frac{r}{2}\right]} & \text{if } z \in \{g^{\left[\frac{r}{2}\right]}, g^{-\left[\frac{r}{2}\right]}\}, \end{cases} \tag{5}$$

with $\alpha_1,\dots,\alpha_{\left[\frac{r}{2}\right]}\in[0,1]$ *and* $\alpha_u\ge T(\alpha_j,\alpha_k)$ *for all:*

$$(u,j,k)\in D_r=\{(x,y,z)\in\left\{1,\dots,\left[\frac{r}{2}\right]\right\}^3\mid x\in\{y+z,z-y,r-y-z\}\}.$$

Proof. $(i)\Rightarrow(ii)$ Let us suppose that μ is a T-subgroup of $G=\langle g\rangle$ so that the properties $G1-G3$ are satisfied. It is therefore necessary that $\mu(e)=1$. In addition, for each $u\in\left\{1,\dots,\left[\frac{r}{2}\right]\right\}$ we can define $\alpha_u:=\mu(g^u)$. Since μ is T-subgroup and $(g^u)^{-1}=g^{-u}$, we can describe μ as in (5).

It only remains to prove that $\alpha_u\ge T(\alpha_j,\alpha_k)$ for all

$$(u,j,k)\in\{(x,y,z)\in\left\{1,\dots,\left[\frac{r}{2}\right]\right\}^3\mid x\in\{y+z,z-y,r-y-z\}\}.$$

We will verify the inequality for each possible value of u, always keeping in mind that $1\le u\le\left[\frac{r}{2}\right]$ and that μ is T-subgroup:

1. If $u=j+k$:

$$\alpha_u=\mu(g^u)=\mu(g^{j+k})=\mu(g^jg^k)\ge T(\mu(g^j),\mu(g^k))=T(\alpha_j,\alpha_k).$$

2. If $u=k-j$:

$$\alpha_u=\mu(g^u)=\mu(g^{k-j})\ge T(\mu(g^k),\mu(g^{-j}))=T(\alpha_j,\alpha_k).$$

3. If $u=r-j-k$:

$$\alpha_u=\mu(g^u)=\mu(g^{r-j-k})=\mu(g^{-j-k})\ge T(\mu(g^{-j}),\mu(g^{-k}))=T(\alpha_j,\alpha_k).$$

Therefore μ must be in the form described in (ii).

$(ii)\Rightarrow(i)$ It suffices to show that μ defined as in (ii) fulfills the T-subgroup properties. $G1$ is trivially satisfied. To derive $G2$ let us take $a\in G\setminus\{e\}$. We can write $a=g^{u_0}$ with $u_0\in\{1,\dots,r-1\}$.

- If $u_0\in\left\{1,\dots,\left[\frac{r}{2}\right]\right\}$, then $\mu(a)=\mu(g^{u_0})=\alpha_{u_0}=\mu(g^{-u_0})=\mu(a^{-1})$.
- If $u_0\in\left\{\left[\frac{r}{2}\right]+1,\dots,r-1\right\}$, then $r-u_0\in\left\{1,\dots,\left[\frac{r}{2}\right]\right\}$ and:

$$\mu(a)=\mu(g^{u_0})=\mu(g^{u_0-r})=\mu(g^{-(r-u_0)})=\alpha_{r-u_0}=\mu_(g^{r-u_0})=\mu(g^{-u_0})=\mu(a^{-1}).$$

Thus, μ satisfies $G2$.

Finally, let us check $G3$. Let $a,b\in G$ and suppose that $a,b,ab\notin\{e\}$ (otherwise $G3$ is directly satisfied). Since:

$$G\setminus\{e\}=\left\{g^1,\dots,g^{r-1}\right\}=\bigcup_{i\in\left\{1,\dots,\left[\frac{r}{2}\right]\right\}}\left\{g^i,g^{-i}\right\},$$

it is clear that there are $u,j,k\in\left\{1,\dots,\left[\frac{r}{2}\right]\right\}$ such that $a\in\left\{g^j,g^{-j}\right\}$, $b\in\left\{g^k,g^{-k}\right\}$ and $ab\in\{g^u,g^{-u}\}\subseteq\left\{g^{j+k},g^{k-j},g^{j-k},g^{-j-k}\right\}$. Furthermore, $g^{r-j-k}=g^{-j-k}=(g^{j+k})^{-1}$ and $g^{j-k}=(g^{k-j})^{-1}$. Two situations are possible:

1. $\alpha_u=\mu(ab)=\mu(g^{j+k})=\mu(g^{r-j-k})$.
2. $\alpha_u=\mu(ab)=\mu(g^{k-j})=\mu(g^{j-k})$.

Note that, given $j,k\in\left\{1,\dots,\left[\frac{r}{2}\right]\right\}$, either $r-j-k\in\left\{1,\dots,\left[\frac{r}{2}\right]\right\}$ or $j+k\in\left\{1,\dots,\left[\frac{r}{2}\right]\right\}$. The same applies to $k-j$ and $j-k$, one of them is automatically included in the set $\left\{1,\dots,\left[\frac{r}{2}\right]\right\}$. Then, $\mu(ab)=\alpha_u\in\left\{\alpha_{j+k},\alpha_{k-j},\alpha_{j-k},\alpha_{r-j-k}\right\}$ and this means that $\mu(ab)=\alpha_{u_0}$ with $u_0\in\{j+k,k-j,j-k,r-j-k\}$. It is clear that either $(u_0,j,k)\in D_r$ or $(u_0,k,j)\in D_r$. Thus, due to the commutativity of T, we have:

$$\mu(ab)=\alpha_{u_0}\ge T(\alpha_j,\alpha_k)=T(\mu(a),\mu(b)).$$

This shows that $G3$ is satisfied and that μ is T-subgroup of G. $\square$

Remark 4.12. In the previous theorem, the use of the integer part of a real number $a\in\mathbb{R}$ avoids duplicating the proof, since it includes the cases where the order of the group is even or odd. In this respect, note also that if the order r of the group is even and g is a generator of the group:

$$g^{\left[\frac{r}{2}\right]}=g^{\frac{r}{2}}=g^{r-\frac{r}{2}}=g^{-\frac{r}{2}}=g^{-\left[\frac{r}{2}\right]}.$$

In the following result we obtain a characterization of the preservation of T-subgroups by aggregation using type-1 domination for cyclic groups of order 4 or 5.

Theorem 4.13. *Let $G = \langle g \rangle$ be a cyclic group with 4 or 5 elements, $\boldsymbol{A}$ an aggregation operator and T a t-norm. The following propositions are equivalent:*

(i) $\boldsymbol{A}$ preserves T-subgroups of G on products.
(ii) $\boldsymbol{A} \gg_1 T$

Proof. $(i) \Rightarrow (ii)$ Let us consider $(x_1, \dots, x_n), (y_1, \dots, y_n) \in [0,1]^n$ such that, for each $i \in \{1, \dots, n\}$ one of these conditions is satisfied:

- $\max\{x_i, y_i\} = 1$ or
- $x_i = y_i$.

We will show that inequality (1) holds.

To prove this, we will choose n T-subgroups $\mu_1, \dots, \mu_n$ and $\boldsymbol{a}, \boldsymbol{b} \in G^n$ so that for each $i \in \{1, \dots, n\}$:

$$x_i = \mu_i(a_i), \quad y_i = \mu_i(b_i) \quad \text{and} \quad \mu_i(a_i b_i) = T(x_i, y_i).$$

Hence, as $\boldsymbol{A}$ preserves T-subgroups of G on products:

$$\begin{aligned} A(T(x_1,y_1),\dots,T(x_n,y_n)) &= A \circ \tilde{\boldsymbol{\mu}}(\boldsymbol{ab}) \geq T(A \circ \tilde{\boldsymbol{\mu}}(\boldsymbol{a}), A \circ \tilde{\boldsymbol{\mu}}(\boldsymbol{b})) \\ &= T(A(x_1,\dots,x_n), A(y_1,\dots,y_n)), \end{aligned} \tag{6}$$

concluding that $\boldsymbol{A} \gg_1 T$.

For this choice, we will consider two situations for each $i \in \{1, \dots, n\}$. Whenever $\max\{x_i, y_i\} = 1$ we will define the T-subgroups:

$$\mu_i(z) = \begin{cases} 1 & \text{if } z = e, \\ \min\{x_i, y_i\} & \text{if } z \neq e. \end{cases}$$

In addition, we will set $a_i \neq e$ and $b_i = e$ if $\min\{x_i, y_i\} \in \{x_i, 1\}$ and $a_i = e$ y $b_i \neq e$ if $\min\{x_i, y_i\} = y_i < 1$. In this case $x_i = \mu_i(a_i)$, $y_i = \mu_i(b_i)$ and $\mu_i(a_i b_i) = \min\{x_i, y_i\} = T(x_i, y_i)$.

Conversely, if $x_i = y_i$, the T-subgroup should be defined as:

$$\mu_i(z) = \begin{cases} 1 & \text{if } z = e, \\ x_i & \text{if } z \in \{g, g^{-1}\}, \\ T(x_i, x_i) & \text{if } z \in \{g^2, g^{-2}\}. \end{cases}$$

Proving that μ_i is indeed T-subgroup of G is straightforward because of Theorem 4.11.

Here, we will take $a_i = b_i = g$ with the aim that $\mu_i(a_i) = \mu_i(b_i) = x_i = y_i$ and $\mu_i(a_i b_i) = T(x_i, x_i) = T(x_i, y_i)$. With this choice, the conditions are in place for (6) to hold and therefore $\boldsymbol{A} \gg_1 T$.

$(ii) \Rightarrow (i)$ Let $n \in \mathbb{N}$ and $\mu_1, \dots, \mu_n$ arbitrary T-subgroups of G. From Theorem 4.11 we have that:

$$\mu_i(z) = \begin{cases} 1 & \text{if } z = e, \\ \alpha_i & \text{if } z \in \{g, g^{-1}\}, \\ \beta_i & \text{if } z \in \{g^2, g^{-2}\}, \end{cases}$$

for some $\alpha_i, \beta_i \in [0,1]$ for each $i \in \{1, \dots, n\}$. Now, given $\boldsymbol{a}, \boldsymbol{b} \in G^n$ we will look for $(x_1, \dots, x_n), (y_1, \dots, y_n) \in [0,1]$ such that either $\max\{x_i, y_i\} = 1$ or $x_i = y_i$. In addition, we choose:

$$x_i \geq \mu_i(a_i), \quad y_i \geq \mu_i(b_i) \quad \text{and} \quad \mu_i(a_i b_i) \geq T(x_i, y_i).$$

Since $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ fulfill the conditions of Definition 3.14, we have that:

$$\begin{aligned} A \circ \tilde{\boldsymbol{\mu}}(\boldsymbol{ab}) &\geq A(T(x_1,y_1),\dots,T(x_n,y_n)) \\ &\geq T(A(x_1,\dots,x_n), A(y_1,\dots,y_n)) \\ &\geq T(A \circ \tilde{\boldsymbol{\mu}}(\boldsymbol{a}), A \circ \tilde{\boldsymbol{\mu}}(\boldsymbol{b})). \end{aligned} \tag{7}$$

To make such a selection, let us consider three cases:

(a) If $\mu_i(a_i b_i) \geq \min\{\mu_i(a_i), \mu_i(b_i)\} = \mu_i(a_i)$ we take:

$$x_i = \mu_i(a_i b_i) \geq \mu_i(a_i) \quad \text{and} \quad y_i = 1 \geq \mu_i(b_i).$$

(b) If $\mu_i(a_i b_i) \geq \min\{\mu_i(a_i), \mu_i(b_i)\} = \mu_i(b_i)$ the choice is:

$$x_i = 1 \geq \mu_i(a_i) \quad \text{and} \quad y_i = \mu_i(a_i b_i) \geq \mu_i(b_i).$$

(c) Finally, we see that if $1 > \min\{\mu_i(a_i), \mu_i(b_i)\} > \mu_i(a_i b_i) \geq T(\mu_i(a_i), \mu_i(b_i))$ then $\mu_i(a_i) = \mu_i(b_i)$. Suppose otherwise that $\mu_i(a_i) \neq \mu_i(b_i)$. Indeed, if this were the case then:

$$\min\{\alpha_i, \beta_i\} = \min\{\mu_i(a_i), \mu_i(b_i)\} > \mu_i(a_i b_i),$$

but $\mu_i(a_i b_i) \in \{\alpha_i, \beta_i\}$ so this leads to a contradiction. Consequently, we will set the values $x_i = y_i = \mu_i(a_i) = \mu_i(b_i)$.

Now we can easily check that, in any case, x_i and y_i satisfy the conditions that we have initially demanded. Therefore, inequality (7) holds and $\boldsymbol{A}$ preserves T subgroups of G on products. □

Corollary 4.14. *Let $G = \langle g \rangle$ be a cyclic group with 4 or 5 elements, $\boldsymbol{A}$ an aggregation operator and T a t-norm. If $\boldsymbol{A} \gg_1 T$ then $\boldsymbol{A}$ preserves T-subgroups on sets.*

Remark 4.15. In Example 4.8 we showed that T_P does not preserve $T_{P/2}$-subgroups of cyclic groups with more than 3 elements on products. In particular, if the order of G is 4 or 5 by Theorem 4.13, $T_P \not\gg_1 T_{P/2}$. Remark 4.8 states that $T_P \gg_0 T_{P/2}$ so this proves that $\gg_0$ and $\gg_1$ are different properties.

4.3. Groups with prime order greater than 5

In this section we will study the necessary and sufficient conditions for an aggregation $\boldsymbol{A}$ to preserve T-subgroups of groups with a prime number of elements greater than or equal to 7. We will see that type-k domination is again key in this case. But first it is necessary to state a lemma that provides some specific T-subgroups defined over cyclic groups. Moreover, it clarifies and shortens the proof of Theorem 4.17.

Lemma 4.16. *Let T be a t-norm and $G = \langle g \rangle$ a group such that $|G| = r > 5$. Given $x, y \in [0,1]$ and $v \in \{2, \ldots, \left[\frac{r}{2}\right] - 1\}$ with $x \geq x_T^{(v-1)} \geq y \geq x_T^{(v)} \geq x_T^{(\left[\frac{r}{2}\right]-1)}$. Then, the fuzzy set:*

$$\mu(z) = \begin{cases} 1 & \text{if } z = e, \\ x_T^{(u)} & \text{if } z \in \{g^u, g^{r-u}\} \text{ with } 1 \leq u < v, \\ y & \text{if } z \in \{g^u, g^{r-u}\} \text{ with } u = v, \\ \alpha_u & \text{if } z \in \{g^u, g^{r-u}\} \text{ with } v+1 \leq u \leq \left[\frac{r}{2}\right] \end{cases} \tag{8}$$

where in the last case α_u can be any number in the interval $[T(x,y), y]$, is a T-subgroup.

Proof. Given $\alpha_u := \mu(g^u)$ for all $u \in \{1, \ldots, \left[\frac{r}{2}\right]\}$, by Theorem 4.11 it is enough to check that $\alpha_u \geq T(\alpha_j, \alpha_k)$ for all:

$$(u, j, k) \in D_r = \{(x, y, z) \in \left\{1, \ldots, \left[\frac{r}{2}\right]\right\}^3 \mid x \in \{y+z, z-y, r-y-z\}\}.$$

First of all, if $x = 1$, then $\alpha_u = 1$ for all $u \in \left\{1, \ldots, \left[\frac{r}{2}\right]\right\}$ since:

$$1 = x_T^{(u)} = x \geq y \geq x_T^{(\left[\frac{r}{2}\right]-1)} = 1.$$

Therefore, $\mu(z) = 1$ for all $z \in G$ and μ is a trivial T-subgroup.

Let us assume that $1 > x \geq x_T^{(u)} \geq x_T^{(\left[\frac{r}{2}\right]-1)}$. We also have that $1 > x \geq y$, so $\alpha_u \neq 1$ for all $u \in \left\{1, \ldots, \left[\frac{r}{2}\right]\right\}$. If $\alpha_u \geq \min\{\alpha_j, \alpha_k\}$, the statement $\alpha_u \geq T(\alpha_j, \alpha_k)$ is straightforward. We will therefore check that when $\min\{\alpha_j, \alpha_k\} > \alpha_u$ we also have $\alpha_u \geq T(\alpha_j, \alpha_k)$. Note that $u \leq j + k$ whenever $u, j, k \in \left\{1, \ldots, \left[\frac{r}{2}\right]\right\}$ with $u \in \{k+j, k-j, r-k-j\}$:

- If $u = j + k$ the inequality is trivially satisfied.
- If $u = k - j$ then $u \leq k \leq k + j$.
- If $u = r - j - k$, by hypothesis $u \leq \left[\frac{r}{2}\right]$. On the one hand, for an odd r, $u \leq \frac{r-1}{2}$ and $u < u + 1 \leq r - u = k + j$. On the other hand, if r is even, then $u \leq \frac{r}{2}$ and $u \leq r - u = k + j$.

Using Lemmas 3.12 and 3.13 we will consider three possible cases:

1. If $1 \leq u < v$, then $\alpha_u = x_T^{(u)}$. Since:

$$1 > \min\{\alpha_j, \alpha_k\} > \alpha_u = x_T^{(u)} \geq x_T^{(v-1)} \geq y,$$

therefore $\alpha_j = x_T^{(j)}$ and $\alpha_k = x_T^{(k)}$. Taking into account that $u \leq j + k$:

$$\alpha_u = x_T^{(u)} \geq x_T^{(j+k)} = T(x_T^{(j)}, x_T^{(k)}) = T(\alpha_j, \alpha_k).$$

2. If $u = v$, we have that $\alpha_u = y$. As before, $\alpha_j = x_T^{(j)}$ and $\alpha_k = x_T^{(k)}$ so we have:

$$\alpha_u = y \geq x_T^{(v)} = x_T^{(u)} \geq x_T^{(j+k)} = T(\alpha_j, \alpha_k).$$

3. If $v + 1 \leq u \leq \left[\frac{r}{2}\right]$, then $\min\{\alpha_j, \alpha_k\} > \alpha_u \in [y, T(x, y)]$. Here again we must distinguish 3 cases:
   (a) If $\alpha_j \leq y$, since $\alpha_k \neq 1$, then $x \geq \alpha_k$ and:

   $$\alpha_u \geq T(x, y) \geq T(\alpha_j, \alpha_k).$$

   (b) If $\alpha_k \leq y$, the result is analogous to point (a).
   (c) If $\alpha_j = x_T^{(j)}$ and $\alpha_k = x_T^{(k)}$. Since $y \geq x_T^{(v)}$:

   $$\alpha_u \geq T(x, y) \geq x_T^{(v+1)} \geq x_T^{(u)} \geq x_T^{(j+k)} = T(\alpha_j, \alpha_k).$$

All the previous discussion shows that $\alpha_u \geq T(\alpha_j, \alpha_k)$ for all $(u, j, k) \in D_r$. Applying Theorem 4.11, we get that the expression given in (8) is a T-subgroup. □

At this stage, we can characterize the aggregation of T-subgroups when the ambient group has prime order p greater than or equal to 7 in terms of type-($\frac{p-3}{2}$) domination.

Theorem 4.17. *Let $G = \langle g \rangle$ be a group with prime order $p \geq 7$, $\boldsymbol{A}$ an aggregation operator and T a t-norm. The following assertions are equivalent:*

(i) $\boldsymbol{A}$ preserves T-subgroups of G on products.
(ii) $\boldsymbol{A} \gg_{\frac{p-3}{2}} T$

Proof. $(i) \Rightarrow (ii)$ Let us suppose that $\boldsymbol{A}$ preserves T-subgroups on products. Let us see that, fixing $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in [0, 1]^n$ such that for each $i \in \{1, \ldots, n\}$ one of the following conditions is fulfilled:

- $\max\{x_i, y_i\} = 1$ or
- $\min\left\{x_i, y_i\right\} \geq \max\left\{x_i, y_i\right\}_T^{(\frac{p-3}{2})}$,

then:

$$A(T(x_1, y_1), \ldots, T(x_n, y_n)) \geq T(A(x_1, \ldots, x_n), A(y_1, \ldots, y_n)).$$

For this purpose we will proceed in a similar way as we did in the proof of Theorem 4.13. That is, we will choose n T-subgroups $\mu_1, \ldots, \mu_n$ and $\boldsymbol{a}, \boldsymbol{b} \in G^n$ such that

$$x_i = \mu_i(a_i), \quad y_i = \mu_i(b_i) \quad \text{and} \quad \mu_i(a_i b_i) = T(x_i, y_i) \quad \text{for all } i \in \{1, \ldots, n\}.$$

We will make this choice according to the relationship between x_i and y_i. If $\max\{x_i, y_i\} = 1$ we will define the T-subgroup:

$$\mu_i(z) = \begin{cases} 1 & \text{if } z = e, \\ \min\{x_i, y_i\} & \text{if } z \neq e \end{cases}$$

as in the proof of Theorem 4.3. Then, we will select:

(a) $a_i \neq e$ and $b_i = e$ if $\min\{x_i, y_i\} \in \{x_i, 1\}$,
(b) $a_i = e$ and $b_i \neq e$ if $\min\{x_i, y_i\} = y_i < 1$.

Otherwise, when $\max\{x_i, y_i\} \neq 1$, there exists $v \in \left\{2, \ldots, \frac{p-3}{2}\right\}$ such that $(M_i)_T^{(v-1)} \geq m_i \geq (M_i)_T^{(v)}$ where:

$$M_i = \max\left\{x_i, y_i\right\} \quad \text{and} \quad m_i = \min\left\{x_i, y_i\right\}.$$

We have this situation because $\max\left\{x_i, y_i\right\} \geq \min\left\{x_i, y_i\right\} \geq (\max\left\{x_i, y_i\right\})_T^{(\frac{p-3}{2})}$ and thus:

$$m_i \in \left[(M_i)_T^{(\frac{p-3}{2})}, M_i\right] = \bigcup_{j\in\{2,\dots,\frac{p-3}{2}\}} \left[(M_i)_T^{(j)}, (M_i)_T^{(j-1)}\right].$$

In this scenario, we will consider the fuzzy set:

$$\mu_i(z) = \begin{cases} 1 & \text{if } z = e, \\ (M_i)_T^{(u)} & \text{if } z \in \{g^u, g^{p-u}\} \text{ with } 1 \le u < v, \\ m_i & \text{if } z \in \{g^u, g^{p-u}\} \text{ with } v \le u \le \frac{p-3}{2}, \\ T(M_i, m_i) & \text{if } z \in \left\{ g^{\frac{p-1}{2}}, g^{\frac{p+1}{2}} \right\}. \end{cases}$$

Lemma 4.16 ensures that μ_i is T-subgroup.

Now, we will select a_i and b_i as follows:

(a) If $M_i = x_i$, we choose $a_i = g$ and $b_i = g^{\frac{p-3}{2}}$,
(b) If $M_i = y_i$, we choose $a_i = g^{\frac{p-3}{2}}$ and $b_i = g$.

In any case $a_i b_i = g^{\frac{p-1}{2}}$ and therefore:

$$x_i = \mu_i(a_i), \quad y_i = \mu_i(b_i) \quad \text{and} \quad \mu_i(a_i b_i) = T(M_i, m_i) = T(x_i, y_i).$$

With this values for $\boldsymbol{a}$ and $\boldsymbol{b}$ and provided that $\boldsymbol{A}$ preserves T-subgroups on products, we have that:

$$\begin{aligned} A(T(x_1,y_1),\dots,T(x_n,y_n)) &= A\circ\tilde{\boldsymbol{\mu}}(\boldsymbol{ab}) \ge T(A\circ\tilde{\boldsymbol{\mu}}(\boldsymbol{a}), A\circ\tilde{\boldsymbol{\mu}}(\boldsymbol{b})) \\ &= T(A(x_1,\dots,x_n), A(y_1,\dots,y_n)). \end{aligned}$$

$(ii) \Rightarrow (i)$ Let us suppose that $\boldsymbol{A} \gg_{\frac{p-3}{2}} T$. Given

$$\boldsymbol{a} = (a_1,\dots,a_n), \boldsymbol{b} = (b_1,\dots,b_n) \in G^n$$

and the T-subgroups $\mu_1,\dots,\mu_n$, we will find $(x_1,\dots,x_n),(y_1,\dots,y_n) \in [0,1]^n$ such that for each $i \in \{1,\dots,n\}$ either $\max\{x_i,y_i\} = 1$ or $\min\{x_i,y_i\} \ge (\max\{x_i,y_i\})_T^{(\frac{p-3}{2})}$. Moreover, we impose that:

$$x_i \ge \mu_i(a_i), \quad y_i \ge \mu_i(b_i) \quad \text{and} \quad \mu_i(a_i b_i) \ge T(x_i, y_i).$$

With all this set, the inequality:

$$\begin{aligned} A\circ\tilde{\boldsymbol{\mu}}(\boldsymbol{ab}) &\ge A(T(x_1,y_1),\dots,T(x_n,y_n)) \\ &\ge T(A(x_1,\dots,x_n), A(y_1,\dots,y_n)) \\ &\ge T(A\circ\tilde{\boldsymbol{\mu}}(\boldsymbol{a}), A\circ\tilde{\boldsymbol{\mu}}(\boldsymbol{b})), \end{aligned} \tag{9}$$

holds and, consequently, $\boldsymbol{A}$ preserves T-subgroups on products. To choose the value of x_i and y_i for each $i \in \{1,\dots,n\}$ we need to consider three different cases:

(a) If $\mu_i(a_i b_i) \ge \min\{\mu_i(a_i), \mu_i(b_i)\} = \mu_i(a_i)$, we choose:

$$x_i = \mu_i(a_i) \quad \text{and} \quad y_i = 1 \ge \mu_i(b_i).$$

(b) If $\mu_i(a_i b_i) \ge \min\{\mu_i(a_i), \mu_i(b_i)\} = \mu_i(b_i)$, the selected values are:

$$x_i = 1 \ge \mu_i(a_i) \quad \text{and} \quad y_i = \mu_i(b_i).$$

(c) If $\min\{\mu_i(a_i), \mu_i(b_i)\} > \mu_i(a_i b_i) \ge T(\mu_i(a_i), \mu_i(b_i))$, then we choose:

$$x_i = \mu_i(a_i) \quad \text{and} \quad y_i = \mu_i(b_i).$$

In all of these cases it follows immediately that $\mu_i(a_i b_i) \ge T(x_i, y_i)$. In addition, for the cases (a) and (b) it is clear that $\max\{x_i, y_i\} = 1$. Special attention should be paid to situation (c) as we will prove that the inequality $\min\{x_i,y_i\} \ge (\max\{x_i,y_i\})_T^{(\frac{p-3}{2})}$ holds when $x_i = \mu_i(a_i)$ and $y_i = \mu_i(b_i)$.

For this purpose, we will take into account the structure of the T-subgroups of this kind of groups. Since $G = \langle g \rangle$ is a prime order cyclic group, we can state that $G = \langle g^s \rangle$ for all $s \in \{1,\dots,p-1\}$. Furthermore, due to Corollary 4.10 only two situations are possible for each $i \in \{1,\dots,n\}$. Either $\mu_i(g^s) = 1$ for all $s \in \{1,\dots,p\}$ or $\mu_i(g^s) \ne 1$ for all $s \in \{1,\dots,p-1\}$. Note that if $\mu_i(g^s) = 1$ for all

$s \in \{1, \ldots, p\}$, then $\mu_i(a_i b_i) \geq \min\{\mu_i(a_i), \mu_i(b_i)\} = T(\mu_i(a_i), \mu_i(b_i))$ and hence we will be in the cases (a) or (b). We focus our attention on $\mu_i(g^s) \neq 1$ for all $s \in \{1, \ldots, p-1\}$. Here, there exists $g_i \in G \setminus \{e\}$ with $1 > \mu_i(g_i) = \max\{\mu_i(g^s) \mid s \in \{1, \ldots, p-1\}\}$. Moreover, as $a_i, b_i \in G = \langle g_i \rangle$, it is clear that there is $s_i \in \{1, \ldots, p-1\}$ so that $\min\{\mu_i(a_i), \mu_i(b_i)\} = \mu_i(g_i^{s_i})$. We show for each possible value of s_i that:

$$\min\{\mu_i(a_i), \mu_i(b_i)\} = \mu_i(g_i^{s_i}) \geq \mu_i(g_i)_T^{(\frac{p-3}{2})} \geq \max\{\mu_i(a_i), \mu_i(b_i)\}_T^{(\frac{p-3}{2})}, \tag{10}$$

where the last inequality is derived from the definition of g_i. The first inequality comes from the following discussion:

- If $1 \leq s_i \leq \frac{p-3}{2}$, then $\mu_i(g_i^{s_i}) \geq \mu_i(g_i)_T^{(s_i)} \geq \mu_i(g_i)_T^{(\frac{p-3}{2})}$.
- If $\frac{p+3}{2} \leq s_i \leq p-1$, then $\mu_i(g_i^{s_i}) = \mu_i(g_i^{p-s_i}) \geq \mu_i(g_i)_T^{(p-s_i)} \geq \mu_i(g_i)_T^{(\frac{p-3}{2})}$.
- If $s_i \in \{\frac{p-1}{2}, \frac{p+1}{2}\}$, since the case where $\mu_i(g_i^{s_i}) = \min\{\mu_i(a_i), \mu_i(b_i)\} > \mu_i(a_i b_i)$ is already settled, the only possibility is that $a_i b_i = g_i^{l_i}$ for some $l_i \notin \{\frac{p-1}{2}, \frac{p+1}{2}\}$. Using the same reasoning as in the two previous cases we can deduce that:

$$\mu_i(g_i^{s_i}) > \mu_i(a_i b_i) \geq \mu_i(g_i)_T^{(\frac{p-3}{2})}.$$

Hence, the expression (10) holds and x_i and y_i have been appropriately selected in order to ensure (9). This means that $\boldsymbol{A}$ preserves T-subgroups of G on products. □

A consequence of the previous result and Proposition 3.2 is that, if $\boldsymbol{A}$ type-$(\frac{p-3}{2})$ dominates T, then $\boldsymbol{A}$ preserves T-subgroups on sets.

Corollary 4.18. *Given a group G with a prime number of elements p, $\boldsymbol{A}$ an aggregation operator and T a t-norm. If $\boldsymbol{A} \gg_{\frac{p-3}{2}} T$ then $\boldsymbol{A}$ preserves T-subgroups of G on sets.*

Whenever $G \notin \mathcal{C}$, Theorem 3.6 shows that domination is a necessary and sufficient condition for an aggregation operator to preserve T-subgroups on products. However, in the present situation the domination requirement must be weakened and replaced by type-k domination. Recall that for the minimum t-norm, domination and type-k domination are equivalent (see point 3 of Remark 3.15). The next example shows that this is not always the case.

Example 4.19. Let us consider the t-norm:

$$T(x, y) = \begin{cases} 0 & \text{if } (x, y) \in]0, 1[^2 \setminus [0.5, 1[^2, \\ \min\{x, y\} & \text{otherwise} \end{cases}$$

and the aggregation operator $\boldsymbol{A}$ by means of the n-ary operators:

$$A(x_1, \ldots x_n) = \begin{cases} 1 & \text{if } x_1 \in [0.5, 1], \\ x_1 & \text{otherwise} \end{cases}$$

with $n \geq 2$. Now we will prove that $\boldsymbol{A} \not\gg T$ although $\boldsymbol{A} \gg_k T$ for all $k \in \mathbb{N}$. In the first place, let us show that the dominance relation is not satisfied. It is enough to take $x_i = y_i = 0.25$ for all $i \in \{2, \ldots, n\}$, $x_1 = 0.75$ and $y_1 = 0.25$:

$$\begin{aligned} A(T(x_1, y_1), \ldots, T(x_n, y_n)) &= A(0, \ldots, 0) = 0 < 0.25 = T(1, 0.25) \\ &= T(A(x_1, \ldots, x_n), A(y_1, \ldots, y_n)). \end{aligned}$$

Now, we will show that $\boldsymbol{A} \gg_k T$ for each $k \geq 0$. Let us check that inequality (1) holds for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in [0, 1]^n$ such that for each $i \in \{1, \ldots, n\}$ one of the following conditions is satisfied:

- $\max\{x_i, y_i\} = 1$.
- $\min\{x_i, y_i\} \geq (\max\{x_i, y_i\})_T^{(k)}$.

We will study the different values of x_1 and y_1. First let us consider that $\max\{x_1, y_1\} = 1$. Without loss of generality, if we suppose that $y_1 = 1$:

$$\begin{aligned} A(T(x_1, y_1), T(x_2, y_2) \ldots, T(x_n, y_n)) &= A(x_1, T(x_2, y_2) \ldots, T(x_n, y_n)) = A(x_1, x_2, \ldots, x_n) \\ &= T(A(x_1, x_2, \ldots, x_n), 1) = T(A(x_1, x_2, \ldots, x_n), A(y_1, y_2, \ldots, y_n)). \end{aligned}$$

These equivalences are deduced from the definition of T and A. The reasoning is analogous if $x_1 = 1$. Hence, let us suppose that $1 > \max\{x_1, y_1\} \geq \min\{x_1, y_1\} \geq (\max\{x_1, y_1\})_T^{(k)}$. Note that, for this t-norm:

$$x_T^{(k)} = \begin{cases} x & \text{if } x \in [0.5, 1], \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the only two possibilities are:

1. $x_1 = y_1 = \max\{x_1, y_1\} = \min\{x_1, y_1\} \geq 0.5$. Here:

$$\begin{aligned} A(T(x_1, y_1), T(x_2, y_2) \dots, T(x_n, y_n)) &= A(x_1, T(x_2, y_2), \dots, T(x_n, y_n)) = 1 = T(1,1) \\ &= T(A(x_1, x_2, \dots, x_n), A(y_1, y_2 \dots, y_n)). \end{aligned}$$

2. $0.5 > \max\{x_1, y_1\} \geq \min\{x_1, y_1\}$. Then:

$$\begin{aligned} A(T(x_1, y_1), T(x_2, y_2) \dots, T(x_n, y_n)) &= A(0, T(x_2, y_2) \dots, T(x_n, y_n)) = 0 = T(x_1, y_1) \\ &= T(A(x_1, x_2, \dots, x_n), A(y_1, y_2, \dots, y_n)). \end{aligned}$$

Therefore, $\boldsymbol{A} \gg_k T$ and $\boldsymbol{A} \not\gg T$ so we can claim that type-k domination and domination are indeed two different properties in general.

4.4. Cyclic groups with order $p^m > 4$, being p a prime and m an integer greater than or equal to 2

In this section, we will show that we need full domination when we consider groups in $\mathcal{C}$ with non-prime order greater than 4.

Theorem 4.20. *Let $G = \langle g \rangle$ be a group with order $p^m > 4$ being p a prime and $m \in \mathbb{N}$ greater than or equal to 2. Let $\boldsymbol{A}$ be an aggregation operator and T a t-norm. The following are equivalent:*

(i) $\boldsymbol{A}$ preserves T-subgroups of G on products.
(ii) $\boldsymbol{A} \gg T$.

Proof. $(ii) \Rightarrow (i)$ is straightforward from Theorem 3.5.

$(i) \Rightarrow (ii)$ Let $(x_1, \dots, x_n), (y_1, \dots, y_n) \in [0,1]^n$ be arbitrary points. Since G is a cyclic group with order p^m, we know that $g^p \in \langle g \rangle$ has order p^{m-1}. Additionally, as $m \geq 2$, $\{e\} \subsetneq \langle g^p \rangle \subsetneq G$. From now on, we will divide the process in two steps. First we will show that the fuzzy sets $\mu_1, \dots, \mu_n$ defined as follows:

$$\mu_i(z) = \begin{cases} 1 & \text{if } z = e, \\ \max\{x_i, y_i\} & \text{if } z \in \langle g^p \rangle \setminus \{e\}, \\ \min\{x_i, y_i\} & \text{if } z \in \{g, g^{-1}\}, \\ T(x_i, y_i) & \text{otherwise} \end{cases} \tag{11}$$

are T-subgroups for each $i \in \{1, \dots, n\}$. We will then use these T-subgroups to prove that A dominates T.

First, it is necessary to check that given $i \in \{1, \dots, n\}$, μ_i is well defined. That is $\langle g^p \rangle \cap \{g, g^{-1}\} = \emptyset$. But whether we assume that $g \in \langle g^p \rangle$ as if we assume $g^{-1} \in \langle g^p \rangle$, we will reach the contradiction $\langle g \rangle \subseteq \langle g^p \rangle \subsetneq \langle g \rangle$. So these T-subgroups are well defined.

We can now check that μ_i satisfies T-subgroup properties. $G1$ y $G2$ are trivially fulfilled. We shall then work on $G3$. Given $z_1, z_2 \in G$, such property is guaranteed if $\mu_i(z_1 z_2) \geq \min\{\mu_i(z_1), \mu_i(z_2)\}$. Let us study the case in which $\min\{\mu_i(z_1), \mu_i(z_2)\} > \mu_i(z_1 z_2)$. This strict inequality together with the construction of μ_i ensures that $\mu_i(z_1), \mu_i(z_2) \in \{1, x_i, y_i\}$ and $\mu_i(z_1 z_2) \in \{x_i, y_i, T(x_i, y_i)\}$. Hence, the only possible situations are the following:

(a) If $z_1 = e$ or $z_2 = e$, the inequality in $G3$ trivially holds.

(b) If $z_1 \notin \langle g^p \rangle$, then $\mu_i(z_1) = \min\{x_i, y_i\}$. Consequently, $\mu_i(z_1 z_2) = T(x_i, y_i)$ and:

$$T(\mu_i(z_1), \mu_i(z_2)) \in \{T(x_i, y_i), T(\min\{x_i, y_i\}, \min\{x_i, y_i\})\}.$$

In all cases, $\mu_i(z_1 z_2) \geq T(\mu_i(z_1), \mu_i(z_2))$. In the case of $z_2 \notin \langle g^p \rangle$, the reasoning is analogous.

(c) If $z_1, z_2 \in \langle g^p \rangle \setminus \{e\}$, then $z_1 z_2 \in \langle g^p \rangle$ and $\mu_i(z_1) = \mu_i(z_2) = \max\{x_i, y_i\}$. Moreover $\mu_i(z_1 z_2) \in \{1, \max\{x_i, y_i\}\}$. Hence, in all the possible situations $\mu_i(z_1 z_2) \geq T(\mu_i(z_1), \mu_i(z_2))$.

With the previous discussion, we have shown that the fuzzy sets $\mu_1, \dots, \mu_n$ are T-subgroups. In the following, it will be also useful to note that $gg^p \notin \langle g^p \rangle \cup \{g, g^{-1}\}$:

- If $gg^p \in \langle g^p \rangle$ then there exists $s \in \mathbb{Z}$ so that $gg^p = (g^p)^s$. Therefore, $g = (g^p)^{s-1} \in \langle g^p \rangle$ which is a contradiction.
- If $gg^p = g$, then $g^p = e$ and this can not happen when $o(g^p) = p^{m-1}$ with $m \geq 2$.
- If $gg^p = g^{-1}$, then $g^{p+2} = e$ and $o(g)$ divides $p+2$. However, if we have $p^m > 4$, it is easy to show that $p^m > p+2$, getting again to a contradiction.

Hence, we can claim that the fuzzy sets $\mu_1, \ldots, \mu_n$ given by the expression (11) are indeed T-subgroups and that $gg^p \notin \langle g^p \rangle \cup \{g, g^{-1}\}$. To finish the proof, let us select $\boldsymbol{a} = (a_1, \ldots, a_n), \boldsymbol{b} = (b_1, \ldots, b_n) \in G^n$ such that for each $i \in \{1, \ldots, n\}$:

- If $\max\{x_i, y_i\} = x_i$, then $a_i = g^p$ and $b_i = g$.
- If $\max\{x_i, y_i\} = y_i$, then $a_i = g$ and $b_i = g^p$.

Thus, $\mu_i(a_i) = x_i$ and $\mu_i(b_i) = y_i$. Furthermore, as $gg^p \notin \langle g^p \rangle \cup \{g, g^{-1}\}$, it follows that $\mu_i(gg^p) = T(x_i, y_i)$. Therefore:

$$A(T(x_1, y_1), \ldots, T(x_n, y_n)) = A \circ \tilde{\boldsymbol{\mu}}(\boldsymbol{ab}) \geq T(A \circ \tilde{\boldsymbol{\mu}}(\boldsymbol{a}), A \circ \tilde{\boldsymbol{\mu}}(\boldsymbol{b}))$$
$$T(A(x_1, \ldots, x_n), A(y_1, \ldots, y_n)),$$

which concludes the proof. □

4.5. Infinite groups

Finally we must analyze the situation where T-subgroups are defined over infinite groups included in $\mathcal{C}$. As mentioned at the beginning of the paper, the only groups with these characteristics are the Prüfer p-groups $\mathbb{Z}(p^\infty)$ with p a prime number. These groups have the characteristic feature that all their elements have finite order p^m for some $m \in \mathbb{N}$. In this infinite groups, in order to ensure the preservation of T-subgroups it is again necessary to demand the domination of $\boldsymbol{A}$ over T as we may have expected given the results in the preceding sections.

Theorem 4.21. *Let $\mathbb{Z}(p^\infty)$ be a Prüfer p-group for some prime number p, $\boldsymbol{A}$ an aggregation operator and T a t-norm. The following statements are equivalent:*

(i) $\boldsymbol{A}$ preserves T-subgroups of $\mathbb{Z}(p^\infty)$ on products.
(ii) $\boldsymbol{A} \gg T$.

Proof. $(ii) \Rightarrow (i)$ It is a direct consequence of Theorem 3.5.

$(i) \Rightarrow (ii)$ As we are working with a Prüfer p-group that contains copies of $\mathbb{Z}_{p^m}$ for each $m \in \mathbb{N}$, we can find an element $g \in \mathbb{Z}(p^\infty)$ such that $o(g) = p^3$. Once we select this element, with a completely analogous reasoning to that of Theorem 4.20, we obtain the result. □

Remark 4.22. Whenever G is a group with 5 or more elements and has a non-trivial proper subgroup, the preservation of T-subgroups of G on products is equivalent to domination. In other words, the only groups for which the preservation of T-subgroups is not always equivalent to domination are those cyclic groups with prime order or those with a cardinality of 4.

5. Conclusions and future work

In this work we have characterized the preservation of T-subgroups on products when the lattice of subgroups of the ambient group is a chain. When this group is the cyclic group of order 4 or a prime order group we have proven that a new property is required in order to characterize preservation of T-subgroups under aggregation. This new property that we have named type-k domination is less restrictive than domination and hence, there will be more aggregation operators preserving T-subgroups in the cases where type-k domination is needed.

Additionally, we have obtained some results that provide valuable information about the structure of T-subgroups when defined over a cyclic group.

Finally, we have obtained some consequences about the preservation of T-subgroups on sets. We are currently conducting a thorough study of aggregation of T-subgroups on sets, which we hope to present soon.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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References

[1] J.M. Anthony, H. Sherwood, Fuzzy groups redefined, J. Appl. Math. Anal. Appl. 69 (1979) 124–130.
[2] J.M. Anthony, H. Sherwood, A characterization of fuzzy subgroups, Fuzzy Sets Syst. 7 (1982) 297–305.
[3] S. Ardanza-Trevijano, M. Chasco, J. Elorza, M. de Natividade, F.J. Talavera, Aggregation of t-subgroups, Fuzzy Sets Syst. 463 (2023) 108390.
[4] C. Bejines, M. Chasco, J. Elorza, Aggregation of fuzzy subgroups, Fuzzy Sets Syst. 418 (2021) 170–184.
[5] C. Bejines, M. Chasco, J. Elorza, J. Recasens, Preserving fuzzy subgroups and indistinguishability operators, Fuzzy Sets Syst. 373 (2019) 164–179.
[6] G. Beliakov, A. Pradera, T. Calvo, Aggregation Functions: A Guide for Practitioners, vol. 221, Springer, 2007.
[7] I. Bloch, On fuzzy distances and their use in image processing under imprecision, Pattern Recognit. 32 (1999) 1873–1895.
[8] D. Boixader, J. Recasens, On the relationship between fuzzy subgroups and indistinguishability operators, Fuzzy Sets Syst. 373 (2019) 149–163.
[9] T. Calvo, A. Kolesárová, M. Komorníková, R. Mesiar, Aggregation operators: properties, classes and construction methods, in: Aggregation Operators, Springer, 2002, pp. 3–104.
[10] T. Calvo, G. Mayor, R. Mesiar, Aggregation Operators: New Trends and Applications, vol. 97, Springer Science & Business Media, 2002.
[11] P.S. Das, Fuzzy groups and level subgroups, J. Math. Anal. Appl. 84 (1981) 264–269.
[12] J. Drewniak, U. Dudziak, Preservation of properties of fuzzy relations during aggregation processes, Kybernetika 43 (2007) 115–132.
[13] J.C. Fodor, M. Roubens, Aggregation and scoring procedures in multicriteria decision making methods, in: [1992 Proceedings] IEEE International Conference on Fuzzy Systems, IEEE, 1992, pp. 1261–1267.
[14] J.C. Fodor, M. Roubens, Fuzzy Preference Modelling and Multicriteria Decision Support, vol. 14, Springer Science & Business Media, 1994.
[15] F. Formato, G. Gerla, L. Scarpati, Fuzzy subgroups and similarities, Soft Comput. 3 (1999) 1–6.
[16] L. Fuchs, Infinite Abelian Groups, Academic Press, 1970.
[17] M. Grabisch, Aggregation Functions, Cambridge University Press, 2009.
[18] E.P. Klement, R. Mesiar, E. Pap, Triangular Norms, vol. 8, Springer Science & Business Media, 2013.
[19] K. Menger, Statistical metrics, Proc. Natl. Acad. Sci. 28 (1942) 535–537.
[20] J. Mordeson, K. Bhutani, A. Rosenfeld, Fuzzy Group Theory, Springer, Berlin, Heidelberg, 2005.
[21] S. Ovchinnikov, Social choice and Lukasiewicz logic, Fuzzy Sets Syst. 43 (1991) 275–289.
[22] T. Pedraza, J. Rodríguez-López, Ó. Valero, Aggregation of fuzzy quasi-metrics, Inf. Sci. 581 (2021) 362–389.
[23] V. Peneva, I. Popchev, Properties of the aggregation operators related with fuzzy relations, Fuzzy Sets Syst. 139 (2003) 615–633.
[24] J. Recasens, Indistinguishability Operators: Modelling Fuzzy Equalities and Fuzzy Equivalence Relations, vol. 260, Springer Science & Business Media, 2010.
[25] R.H. Reiser, B. Bedregal, M. Baczyński, Aggregating fuzzy implications, Inf. Sci. 253 (2013) 126–146.
[26] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35 (1971) 512–517.
[27] S. Saminger, R. Mesiar, U. Bodenhofer, Domination of aggregation operators and preservation of transitivity, Int. J. Uncertain. Fuzziness Knowl.-Based Syst. 10 (2002) 11–35.
[28] I. Silambarasan, Fermatean fuzzy subgroups, J. Int. Math. Virtual Inst. 11 (2021) 1–16.
[29] P. Su, Q. Shen, T. Chen, C. Shang, Ordered weighted aggregation of fuzzy similarity relations and its application to detecting water treatment plant malfunction, Eng. Appl. Artif. Intell. 66 (2017) 17–29.