Transport in Porous Media **16:** 253–261, 1994.
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Capillary Ascension in Porous Media: A Scaling Law

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(Received: 7 June 1993; in final form: 28 December 1993)

Abstract. A relationship between the microstructure of a porous medium and the height of ascension of a fluid in the material is proposed. The porous medium is modelled by means of a square lattice randomly filled with matter. A scaling law is derived from a numerical simulation. An analytical solution is also obtained and compared with the numerical results.

Key words: capillarity, percolation, Monte Carlo procedure, mean-field approximation.

1. Introduction

Macroscopic fluid transport through porous media is usually modelled by means of balance equations which do not contain the capillary term. The most famous constitutive equation expressing mass transport in porous media is that of Darcy relating the pressure gradient to the seepage velocity

$$\nu = -\frac{1}{\mu}\mathsf{K} \cdot \nabla p, \tag{1}$$

ν is the seepage velocity, p the pressure, μ the dynamic viscosity of the fluid, and K the permeability tensor providing an average image of the internal structure of the porous media. Darcy's law has received some theoretical background (Slattery, 1972; Whitaker, 1986; Adler, 1992), and has been improved some years ago by Brinkman (1947) and Forcheimer (Joseph *et al.*, 1982). However, for small values of the seepage velocity, capillary effects become dominant. These effects are conveniently described by means of the dimensionless capillary number

$$C = \frac{\mu\nu}{\gamma}, \tag{2}$$

wherein ν and γ are the modulus of the average velocity and surface tension, respectively. For $C \ll 1$, the flow is quasi-static and capillarity is the single relevant effect (Chandler *et al.*, 1982).

The practical importance of this capillary flow in geological and industrial areas is well known; for instance, it is the flow that occurs in the capillary pump which is responsible for fluid circulation in caloducs (heat pipes). Another example is found in everyday life when the bottom of a piece of sugar is put in contact with

the fluid surface of a cup of tea or coffee, the rise of the fluid in the sugar due to the capillary force is clearly visible.

With regard to capillary ascension, an important law is that of Jurin relating the height of ascension of a fluid inside an open cylindrical capillary tube to its radius, and is given by

$$h = \frac{2\gamma \cos\theta}{\rho g r}, \tag{3}$$

where h is the fluid height, ρ the fluid density, θ the wettability angle, r the radius of the tube, and g the acceleration of gravity. The fluid rising is due to surface tension at the interface between the fluid and its environment, say air, and results in the apparition of a meniscus. The wettability angle depends on the fluid and solid properties. The determination of this wettability angle is a crucial problem and requires a description of the fluid-solid interaction.

The purpose of the present analysis is to generalize the Jurin law for porous media. In particular, a relation between the microstructure of the porous media and the height of ascension of the fluid will be established. In Section 2, a numerical simulation based on a Monte Carlo procedure is proposed. In Section 3, the principal features of the model are outlined. The mean-field approach will be developed in Section 4 and final comments are found in Section 5.

2. Model Description and Numerical Simulation

The porous media is modelled by a square lattice of 200×200 square cells randomly filled with a solid material. The concentration of void space (porosity) is denoted by p, thus $1 - p$ is the probability of finding a cell filled with solid material. In Figure 1, dark blocks represent cells filled with matter while white zones refer to void regions. The physical experiments runs as follows: the bottom of the porous structure is put in contact with the upper surface of a fluid; because of the capillary forces, the fluid flows through the interstices of the porous structure. After a while, the fluid reaches an equilibrium: capillary forces are balanced by gravity forces. Our objective is to determine the maximum height reached by the fluid. This height can be converted into an equivalent capillary pressure drop by the hydrostatic relation $\Delta p = \rho g \Delta h$.

The numerical simulation is directly inspired from the physical experiment just described. The first step consists of filling the void cells of the first row of the array; the next step will take place according to the following rules: the fluid is allowed to propagate through the structure laterally and downward if the void cells are in contact with an already filled cell. By virtue of (3) the fluid will flow upwards with the additional condition that

$$h \leqslant \frac{2\gamma \cos\theta}{\rho g r}. \tag{4}$$

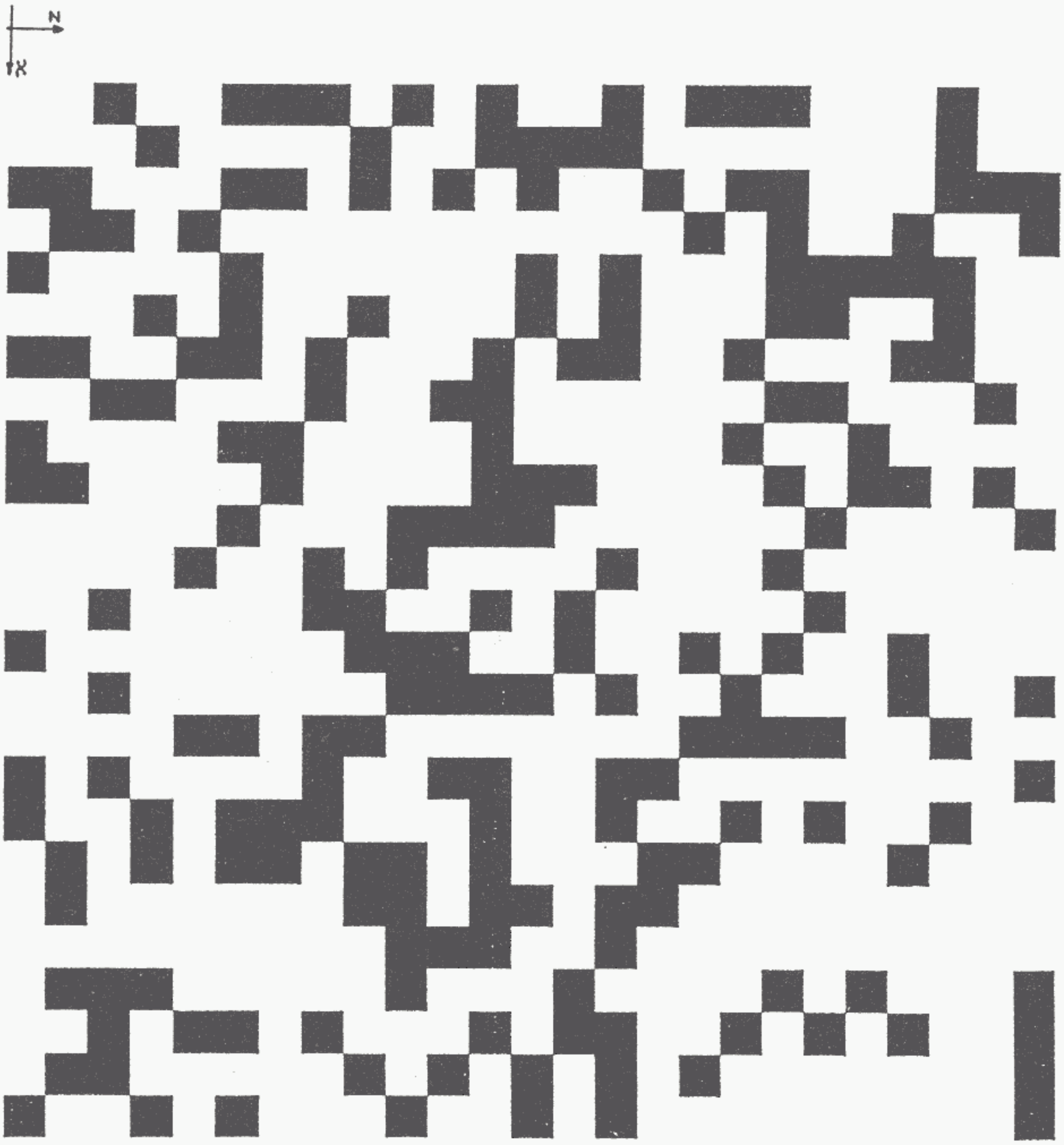

Fig. 1. Square lattice (25×25), randomly filled with material (black pixels), the density of void (white pixels) is $p = 0.65$ (porosity).

Introducing the characteristic dimensions R of the grains of the porous medium, i.e. the dimension of a lattice cell, and the dimensionless quantities $\tilde{h} = h/R$, and $s = r/R$, expression (4) can be written in the form

$$\tilde{h} \leqslant \frac{a}{s}, \tag{5}$$

with

$$a = \frac{2\gamma \cos\theta}{R^2 \rho g}, \tag{6}$$

a is a non-dimensional parameter directly linked to the capillarity; it is related to the inverse of the Bond number which measures the ratio of the buoyancy force and the capillary force.

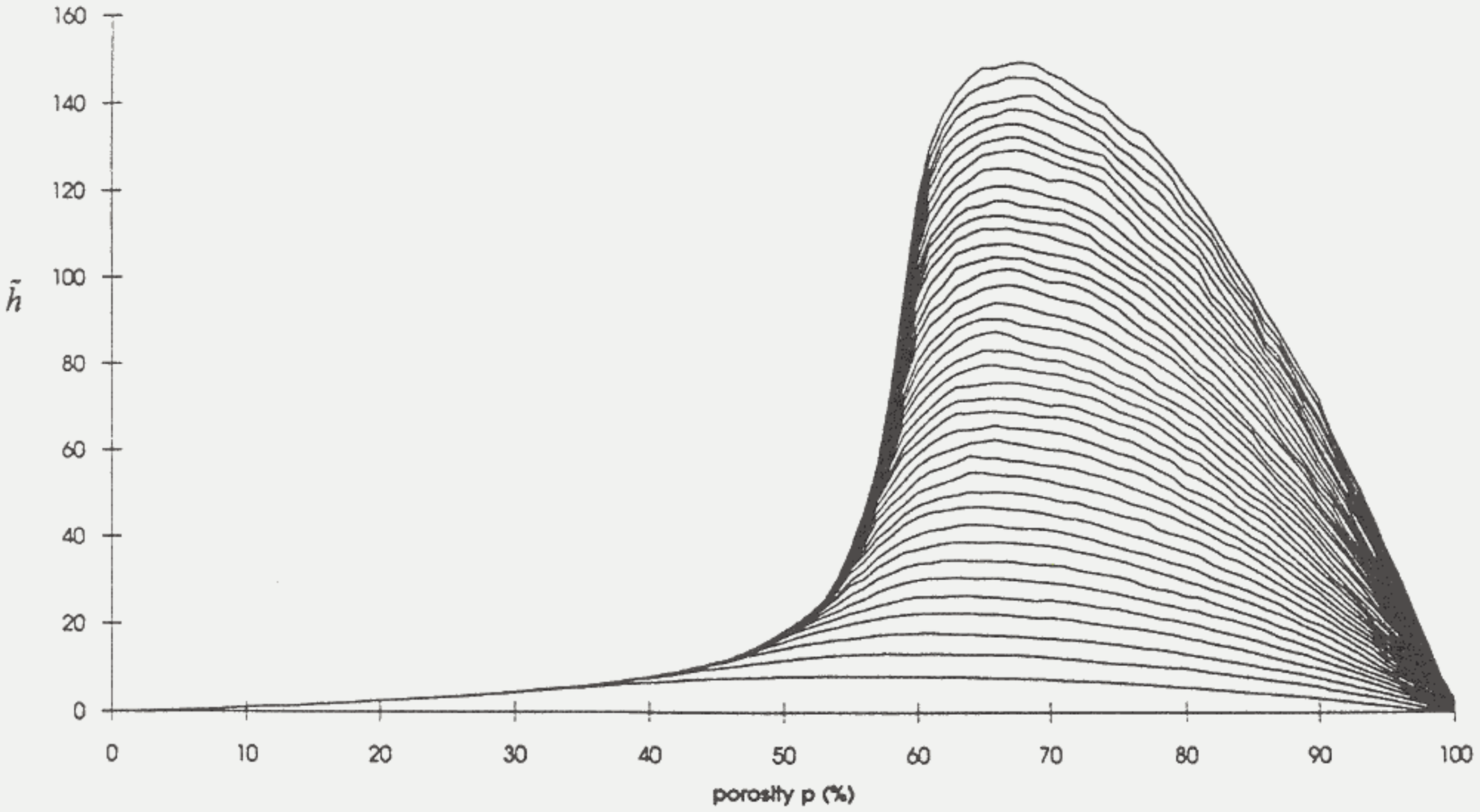

Fig. 2. Height of ascension to $\tilde{h}$ versus porosity p for 40 values of the capillarity parameter (the lowest curve corresponds to $a = 25$, the highest one to $a = 1000$, the steps are $\Delta a = 25$).

Although expression (5) has been derived for a capillary tube, it is now used as a condition giving the maximum allowable height of the fluid in the lattice; in this case, s in (5) represents the number of void cells simply connected in the horizontal row to which pertains the void cell concerned by the condition (5). The simulation process stops when the fluid has filled all the allowed void cells. The simulation described here is equivalent to a percolation problem with constraints (Stauffer, 1985). Concerning the lateral boundary conditions, we choose (Chandler *et al.*, 1982) a periodic structure in order to avoid the side effects becoming dominant.

The results are displayed in Figure 2 where $\tilde{h}$ is compared against the porosity p for different values of the parameter a. The simulation was performed several times for the same porosity in order to avoid statistical fluctuations. It is known that these are important in percolation problems, particularly in the critical zone, i.e. at the maximum of the curves. Figure 2 clearly exhibits two different regions: at low porosity the curves are independent of the parameter a, because the flow is quickly stopped due to the high concentration of the solid matrix; in contrast, at high porosity, the capillarity plays an important role and the curves are separated from each other. In the limiting case $a \gg 1$, one recovers the results of the classical percolation theory without constraint.

3. Scaling Laws and Illustrations

Figure 2 indicates that, for each fixed value of the parameter a, there exists an optimal porosity for which $\tilde{h}$ is maximum. Figure 3 represents this optimal porosity

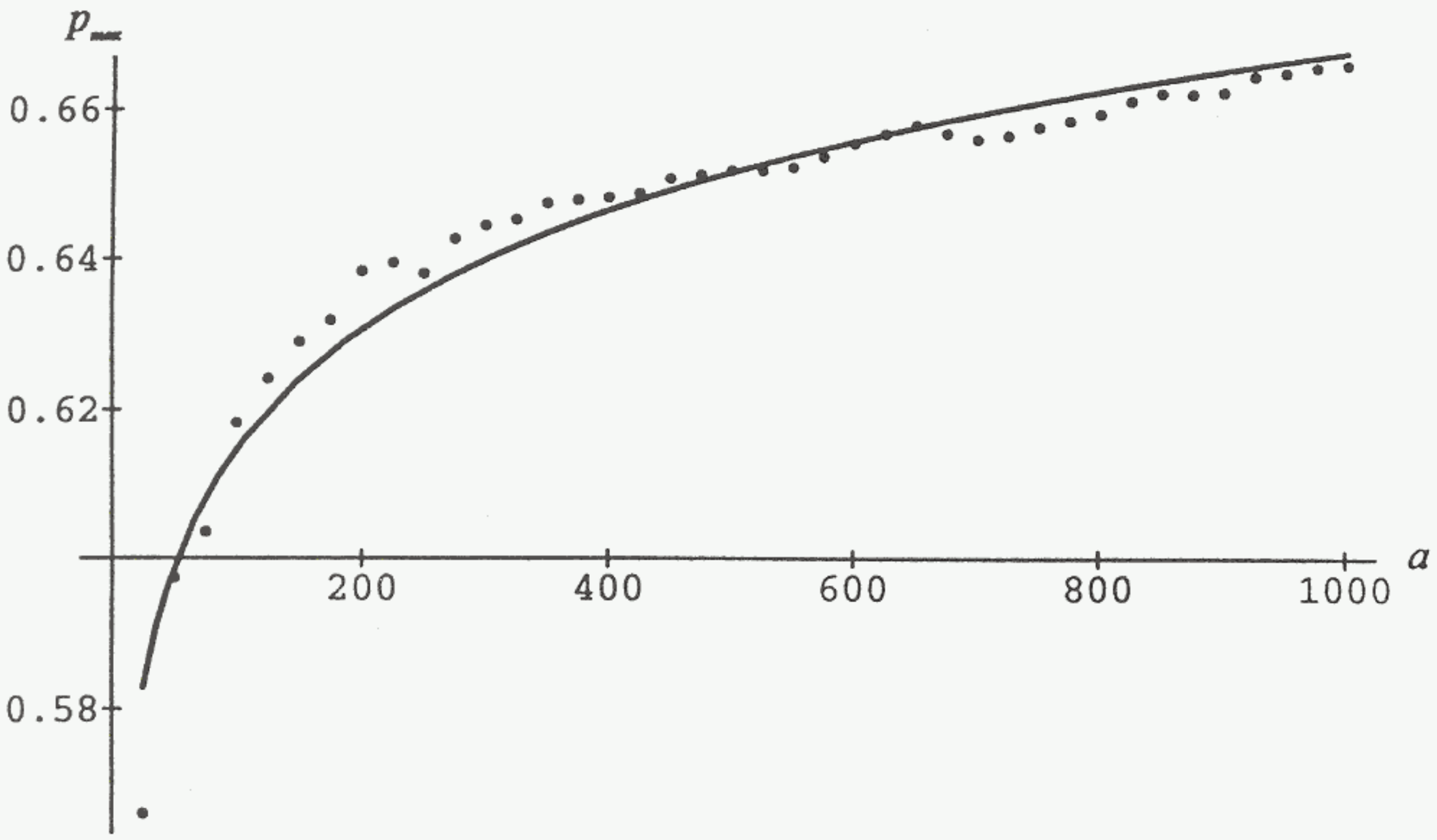

Fig. 3. Optimal porosity $p_{\max}$ versus capillarity a.

$p_{\max}$ versus a, a fitting of these points show that they satisfy an empirical law of the form:

$$p_{\max} = 0.509 + 0.023 \ln a. \tag{7}$$

Similarly, one can construct another scaling law for the maximum value of $\tilde{h}$ versus a. The results are given in Figure 4 and can be analytically expressed as

$$\tilde{h}_{\max} = 0.56 a^{0.805}. \tag{8}$$

It is found that, for a sample of dimensions 200×200, the maximum fluid ascension for $a = 10^3$ is equal to about 150 elementary cells ($\tilde{h} \sim 150$); this means that the real dimensional height of ascension is about 150 times the characteristic size of the grain R of the porous medium ($h \sim 150R$). Experiments by Winslow and Liu (1990) show that, for concrete, the size of the pores is of the order of magnitude of one μm. As a consequence, the present simulation should predict heights of ascension of the order of only 200 μm, which is an unrealistic small value. This means that the finest structure of the porous media requires to be represented by a sample containing many more than 200×200 cells.

To extend the range of application of our analysis, it is assumed that the similitude law (8) remains valid for samples of dimensions much larger than 200×200. A simple example will illustrate the procedure. Assume that the flowing fluid is water (taken to be perfectly wetting: $\cos\theta = 1$) and that the characteristic dimension R of the grains is 1 μm, it follows that

$$R = 10^{-6}\,\mathrm{m}, \quad \tilde{R} = 1\,\mathrm{cell}, \quad \frac{2\gamma}{\rho g} = 1.4 \times 10^{-5}\,\mathrm{m}^2, \quad a = 1.4 \times 10^7.$$

Fig. 4. Scaling law of the maximal height of ascension $\tilde{h}_{\max}$ versus the capillarity a.

From (8), it is then found that

$$\tilde{h} = 3.18 \times 10^5 \text{ cells} \quad \text{or} \quad h = 0.318\,\text{m}.$$

To have an idea of the order of magnitude of the relevant quantity, it is interesting to compare this result with the height of ascension in a capillary tube whose radius is equal to 1 μm. According to Jurin's law (3), one has

$$h = \frac{1.4 \times 10^{-5}}{10^{-6}} = 14\,\text{m}.$$

This example shows that the height of ascension in a porous media obtained from (8) is, in this case, approximately 44 times lower than in a cylindrical capillary tube with a radius of the same size as the dimensions of the grains of the porous media. An examination of the laws (3) and (8) shows that, for the Jurin formula (3), the height of ascension h varies like r^{-1}, where r is the radius of the capillary tube, whereas for the porous media, the functional dependence between the height of ascension and the grain size is $R^{-0.61}$. Of course, these considerations rest on the assumption that the scaling law (8) is applicable for high values of the parameter a. To verify this assumption, simulations with larger lattices have been performed. It was checked that expression (8) is still valid for lattices formed with 400×400 cells. For larger lattices the simulation was not done due to the increase of the computational cost.

4. Mean Field Approach

To complete the analysis, we now re-examine the problem within a mean field context (Bruggeman, 1935; Ronis, 1986; Thovert *et al.*, 1990). Our objective is to express the height of ascension $\tilde{h}$ as a function of the porosity p.

We consider the two-dimensional square lattice as a superposition of one-dimensional rows, for which analytical results are available. The width s of the voids has a statistical distribution given by (Stauffer 1985):

$$P(s) = \begin{cases} 1-p, & \text{if } s = 0, \\ sp^s(1-p)^2, & \text{if } s \in N^0. \end{cases} \tag{9}$$

$P(s)$ is the probability of finding a void of width s by randomly selecting a cell among the row. The average of s is given by

$$\langle s \rangle = \sum_s^\infty sP(s) = \sum_s^\infty s^2 p^s (1-p)^2 = \frac{p(1+p)}{1-p}. \tag{10}$$

It should be noticed that $\langle s \rangle$ is a monotonically increasing function of p and that $\langle s \rangle = 1$ for $p = \sqrt{2} - 1 = 0.414$. For porosity p larger than this value, the average width becomes larger than one, which means that this limiting value corresponds to a percolation threshold for the lattice. This value should be compared with the actual percolation threshold for this square lattice, which is equal to 0.59.

The mean field approach mimics the numerical simulation but only on average. It proceeds as follows. First, the void cells occupying the bottom of the lattice are filled by the fluid so that, at level one, the number of cells filled by the fluid is Np (where N is the width of the array). Afterwards, fluid ascension will be refrained by capillarity effect. Denote by $s_c = \text{int}(a/\tilde{h})$, the maximum integer width permissible at level $\tilde{h}$. Since the summation in expression (10) cannot exceed s_c, formula (10) becomes

$$\begin{aligned} \langle s \rangle = \sum_{s=0}^{s_c} s^2 p^s (1-p)^2 = p[s_c^2 p^{s_c+2} - (2s_c^2 + 2s_c - 1)p^{s_c+1} \\ +(s_c+1)^2 p^{s_c} - p - 1]/(p-1). \end{aligned} \tag{11}$$

In view of (11), the number of cells filled by the fluid at level two is given by

$$Np\langle s \rangle_2 \tag{12}$$

where the subscript 2 means that $\langle s \rangle$ is calculated at $\tilde{h}$ equal 2. Repeating this procedure until the number of fluid cells become less than one, one obtains the value of the maximum elevation $\tilde{h}_{\max}$ attained by the fluid from

$$Np \prod_{i=2}^{\tilde{h}_{\max}} \langle s \rangle_i = 1. \tag{13}$$

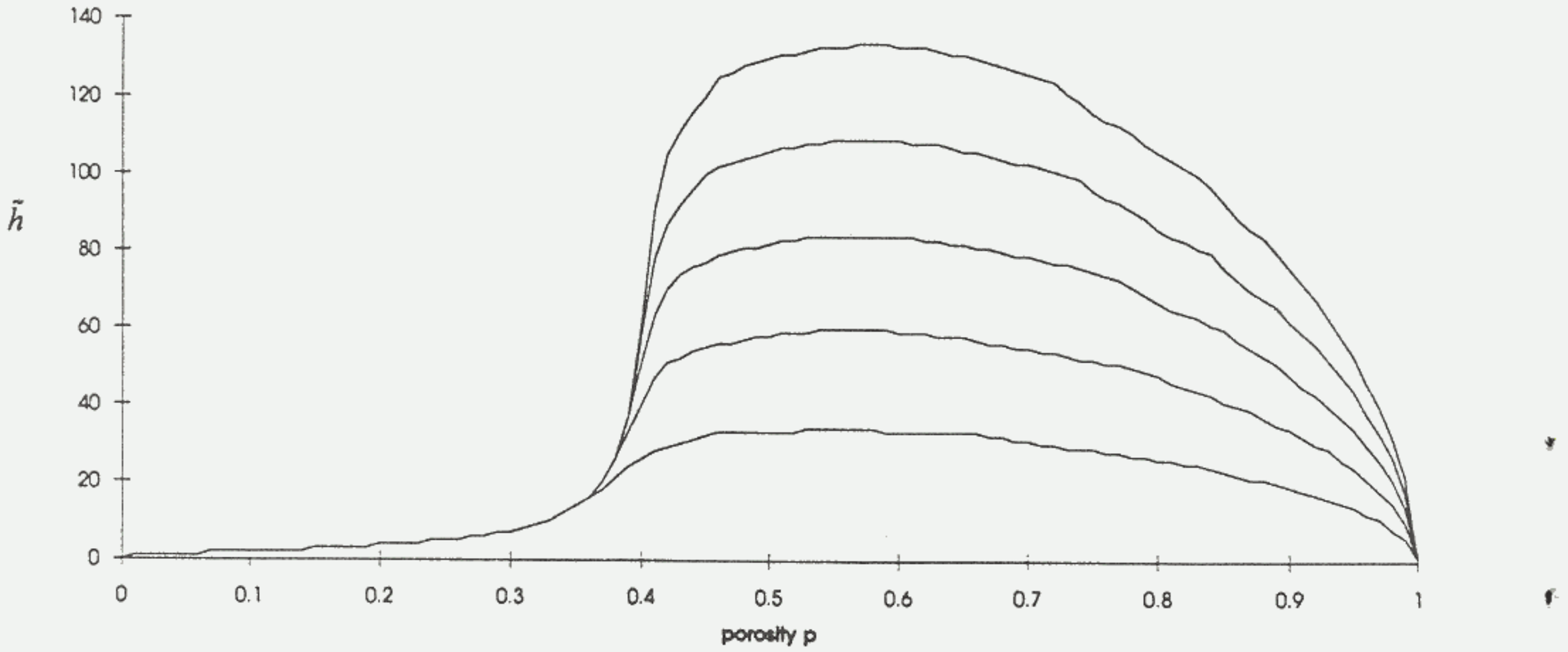

Fig. 5. Mean field approach showing the height of ascension $\tilde{h}$ versus p (porosity) for five values of the capillarity parameter ($a = 100 \rightarrow 500$, step 100).

Figure 5 reproduces the results of this statistical approach for several values of the parameter a. It is observed that only a qualitative agreement is achieved between the numerical simulation (see Figure 2) and the mean-field technique (see Figure 5). The results predicted by the mean-field approach overestimate the numerical one by about 50%. The low quality of the statistical model originates in the fact that no exact solution is available for the two-dimensional percolation problem and that our assumption to modelize a two-dimensional lattice as a superposition of one-dimensional systems appears to be too crude. We have tried to improve the results by introducing perturbative techniques. However, it is well known that perturbative methods applied to mean field theories do not necessarily result in improvements. This is confirmed by the present work as our efforts to obtain better results failed. For this reason, we shall not enter into the details of these techniques.

5. Final Comments

The problem of capillary ascension in a random porous media is examined: a scaling law relating the capillarity and the height of ascension is deduced from numerical simulations. A mean-field approach is also developed, but the corresponding results are only qualitatively acceptable. Furthermore, this approach does not allow us to evaluate the critical exponent of the scaling law. It is also expected that the shape of the porous media will deeply influence the fluid ascension and therefore a shape parameter should be incorporated into the ascension law. Although the experimental measure of the porosity is relatively easy, this is no longer true if, it is wished to determine the internal structure of the porous media (e.g. tortuosity, moments of the grain size distribution) which is not an obvious task. The problem of the wettability angle remains also pendent and is of great concern in relation to the above treated

problem. Concerning the present model, an immediate improvement would consist of using a tri-dimensional lattice rather than a two-dimensional one.

Acknowledgements

The authors wish to thank Professor F. Brouers for fruitful discussions about the percolation theory and Ir. P. Parmentier for his help in the numerical simulation. This text presents research results of the Belgian Interuniversity Poles of Attraction (PAI No. 29) initiated by the Belgian State, Prime Minister's Office, Science Policy Programming.

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